Received: 31 May 2024 | Accepted: 25 September 2024

DOI: 10.1111/sapm.12771

STUDIES IN APPLIED MATHEMATICS WILEY

SPECIAL ISSUE ARTICLE

Recent Advances in the Analysis and Simulation of Compressible Flow Problems: The 75th Anniversary of the Landmark Report by Lagerstrom, Cole, & Trilling (1949)

Existence of solutions to k-Wave models of nonlinear ultrasound propagation in biological tissue

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Funding information
Engineering and Physical Sciences Research Council, Grant/Award Numbers: EP/W029324/1, EP/T014369/1; Austrian Science Fund, Grant/Award Number: 10.55776/P36318

Abstract

We investigate models for nonlinear ultrasound propagation in soft biological tissue based on the one that serves as the core for the software package k-Wave. The systems are solved for the acoustic particle velocity, mass density, and acoustic pressure and involve a fractional absorption operator. We first consider a system that incorporates additional viscosity in the equation for momentum conservation. By constructing a Galerkin approximation procedure, we prove the local existence of its solutions. In view of inverse problems arising from imaging tasks, the theory allows for the variable background mass density, speed of sound, and the nonlinearity parameter in the systems. Second, under stronger conditions on the data, we take the vanishing viscosity limit of the problem, thereby rigorously establishing the existence of solutions for the limiting system as well.

KEYWORDS
fractional Laplacian, k-Wave, local existence, ultrasound modeling

1 INTRODUCTION

Ultrasound waves propagating in soft biological tissue, even at the intensities used in biomedical imaging applications, can undergo noticeable nonlinear distortion. At higher intensities still, such as are used in therapeutic medical applications, the effect of the nonlinearities can be very significant. Several scientific software packages have therefore been developed for modeling nonlinear propagation in biological tissue.[1] Here, the system of equations that are the basis for one of those packages, k-Wave,[2,3] will be analyzed. It is given in terms of the acoustic particle velocity $\boldsymbol{u}$, mass density ρ, and acoustic pressure p by the following set of equations:

$$\begin{aligned}&\text{linear momentum conservation:} && \rho_0 \boldsymbol{u}_t + \nabla p = \boldsymbol{f},\\ &\text{mass conservation:} && \rho_t + (2\rho + \rho_0)\,\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\rho_0 = 0,\\ &\text{pressure-density relation:} && p - c_0^2\Big(\rho + \boldsymbol{d}\cdot\nabla\rho_0 + \frac{B}{2A}\frac{\rho^2}{\rho_0} - \widetilde{L}\rho\Big) = 0,\end{aligned} \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{d}_t$; see Refs. [2, system (10)] and [4, system (1)]. The operator $\widetilde{L}$ accounts for absorption and dispersion. It is defined by

$$\widetilde{L}\rho = 2\alpha_0\left(-c_0^{y-1}(-\Delta)^{\frac{y}{2}-1}\rho_t + c_0^y \tan\left(\frac{\pi y}{2}\right)(-\Delta)^{\frac{y+1}{2}-1}\rho\right) \tag{2}$$

with $y \in (1,3)$ and $\alpha_0 > 0$; see Ref. [4, eq. (3)]. In human tissue, typically $y \in (1,2]$. The quantities ρ_0, c_0, and $\frac{B}{A}$ in this system are the background mass density, isentropic sound speed, and nonlinearity parameter, respectively.

In k-Wave, these equations are discretized using a pseudo-spectral time domain (PSTD) time-stepping scheme with a dispersion correcting factor applied in the spatial Fourier domain. The particular form of the absorption/dispersion term in (1) was chosen both because the resulting absorption depends on frequency according to a power law, as empirically observed in many tissue types, and because it is memory-efficient when implemented using a PSTD scheme.

1.1 Numerical example

In the spirit of motivation for the study of system (1), a simple numerical example, computed using k-Wave, will be given here. With ultrasound tomography in mind, this example shows that for a fixed number of sources and detectors, more independent data can be obtained when nonlinear effects are included than in the linear case. Specifically, inspecting the singular value spectrum of a set of simulated measurements shows that when pairs of sources are used simultaneously in the nonlinear regime, the resulting measured signals are not just linear combinations of the signals measured with the individual sources alone, as they are in the linear case. Figure 1 (left) shows a ring array of eight equally spaced transducer elements that all act as detectors, and four of which (shown in white) also act as sources, surrounding a region with a heterogeneous sound speed.[5] All other material properties were chosen to be homogeneous: mass density $\rho_0 = 1000\,\mathrm{kg/m^3}$, absorption coefficient $\alpha = \alpha_0 f^y$ where $\alpha_0 = 0.5$ dB/cm/MHz^y, $y = 1.5$, $f = 0.25$ MHz is the frequency, $B/A = 7$ is the acoustic nonlinearity parameter, and the source acoustic pressure is 5 MPa. Simulations were conducted in both the nonlinear and linear regimes (i.e., no nonlinear terms

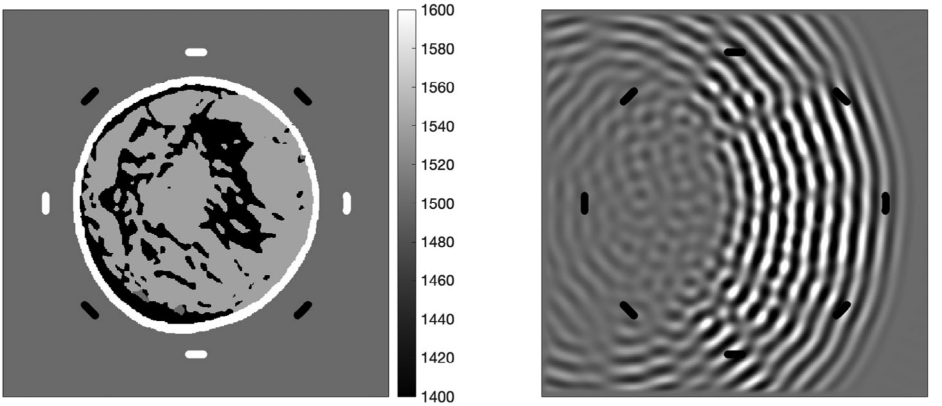

FIGURE 1 Left: Set-up of the numerical example, showing the sound speed map (m/s) and the positions of the transducers (white: sources and detectors, black: detectors only). Right: Snapshot of the field from the leftmost transducer acting as a source.

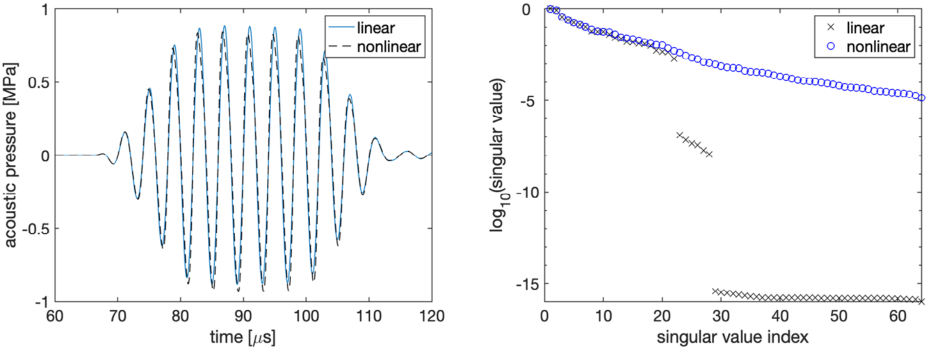

FIGURE 2 Left: Examples of linear and nonlinear time series. Right: Singular value spectrum of the linear and nonlinear data.

included in the equations, equivalent to using a low source amplitude source). For each simulation, the transducers acting as sources were driven with a single-frequency sinusoidal wave, and acoustic pressure time series were detected at all other transducers. When acting as a source, a transducer does not also act as a detector, so in both the linear and nonlinear cases, 64 time series were measured: 28 using single sources (4 sources x 7 detectors), and 36 using pairs of sources driven simultaneously (6 pairs of sources × 6 detectors). Figure 1 (right) shows a snapshot of the acoustic pressure field emitted from the leftmost transducer. Figure 2 (left) shows examples of measured time series in both the linear and nonlinear cases, showing characteristic wave steepening due to the nonlinearity increasing the wave speed at the peaks of the wave and decreasing it at the troughs. All 64 time series measured in the linear case were stacked into a matrix and the singular values of that data matrix were computed. This was also done in the nonlinear case. The singular value spectra, normalized to the largest singular value, are plotted in Figure 2 (right). The cliff-edge after the 28th singular value in the linear case indicates that the data obtained using pairs of sources are merely linear combinations of the data obtained using single sources. This is not the case in the nonlinear regime. While this example may be interesting, we note that it does not prove—or indicate the extent to which—the data carry additional information about the material properties, the estimation of which is the ultimate goal of ultrasound tomography.

1.2 | Main contributions

The main aim of this work is to gain rigorous understanding of the systems of the form in (1) with possible additional viscosity included in the momentum balance equation. Throughout, we assume that $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$, is a bounded domain that is $C^{1,1}$ regular or Lipschitz regular and convex. In view of inverse problems arising from imaging tasks, we are particularly interested in allowing $\frac{B}{A}$, c_0, and ρ_0 in (1) to depend on x in this order of importance, that is, the simplification $\rho_0 \equiv const.$ is the least restrictive one. With this in mind, we can rewrite the mass conservation in terms of $\sigma = \dfrac{\rho}{\rho_0}$ as follows:

$$\sigma_t + (1 + 2\sigma)\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho_0 = 0.$$

To simplify the analysis, we supplement the system with the following homogeneous boundary conditions:

$$\nu \cdot \boldsymbol{u} = 0, \qquad \nu \cdot \nabla\sigma = 0 \ \text{ on } \partial\Omega, \tag{3}$$

where ν is the outer unit normal vector at the boundary $\partial\Omega$, as well as the initial velocity and density data

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{d}(0) = \boldsymbol{d}_0, \qquad \sigma(0) = \sigma_0. \tag{4}$$

Then $\boldsymbol{d} = \mathrm{I}_t\boldsymbol{u} + \boldsymbol{d}_0$, where $\mathrm{I}_t\boldsymbol{u} = \int_0^t \boldsymbol{u}(s)\,\mathrm{d}s$ for $t \in [0, T]$. By taking into account a viscosity term in the momentum balance in (1) and rearranging the terms, we arrive at the following system for $(\boldsymbol{u}, \sigma, p)$:

$$\begin{cases} (\mathrm{mo}^{\mu}) & \rho_0\boldsymbol{u}_t + \nabla p - \mu\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{f}, \\ (\mathrm{ma}) & \sigma_t + a(\sigma)\nabla\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\nabla\ln\rho_0 := \mathrm{g}(\boldsymbol{u}), \\ (\mathrm{pd}) & p - c_0^2\rho_0 b(\sigma)\,\sigma + L\sigma = c_0^2\boldsymbol{d}\cdot\nabla\rho_0 = c_0^2(\mathrm{I}_t\boldsymbol{u} + \boldsymbol{d}_0)\cdot\nabla\rho_0 := h(\boldsymbol{u}), \end{cases} \tag{5}$$

with a *modified* absorption operator

$$L\sigma = -2\alpha_0(-\Delta_{1/\rho_0})^{-1}\left[\tau(-\Delta)^{\frac{y}{2}}\sigma_t + \eta(-\Delta)^{\frac{y+1}{2}}\sigma\right], \quad \tau, \eta > 0, \tag{6}$$

[1]where $-\Delta = -\Delta_N$ denotes the homogeneous Neumann–Laplacian and

$$\Delta_{1/\rho_0} v := \nabla\cdot\left(\frac{1}{\rho_0}\nabla v\right).$$

[1] Note that our analysis could also handle the choice (2), however at the cost of involving higher order commutators of the coefficients ρ_0 and c_0 and thus having to impose higher smoothness on them.

We have introduced the following short-hand notation in (5):

$$a(\sigma) = 1 + 2\sigma, \quad b(\sigma) = 1 + \frac{B}{2A}\sigma \tag{7}$$

and

$$g(\boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla \ln \rho_0, \qquad h(\boldsymbol{u}) = c_0^2 \boldsymbol{d} \cdot \nabla \rho_0 = c_0^2 (\mathrm{I}_t \boldsymbol{u} + \boldsymbol{d}_0) \cdot \nabla \rho_0. \tag{8}$$

We have chosen $-\mu\nabla(\nabla \cdot \boldsymbol{u})$ for the viscosity term with $\mu > 0$ (which is equal to $-\mu\Delta\boldsymbol{u}$ for irrotational $\boldsymbol{u}$) since it allows us to make use of cancellations below without having to impose the equation $\nabla \times \boldsymbol{u} = 0$ as a further partial differential equation (PDE).

The main contributions of the remaining of the work pertain to the analysis of the system in (5); in particular, we establish existence of its solutions in Theorem 1 using a Galerkin-based framework. Additionally, under the assumption that $g = h = 0$, we conduct analysis in the vanishing viscosity limit $\mu \searrow 0$ as a way of relating system (5) to system (1) with the absorption operator (6). This result is contained in Theorem 2 below.

To the best of our knowledge, systems of the form in (5) with fractional absorption have not been studied so far in a rigorous manner. In contrast, rigorous techniques for single-equations models in nonlinear acoustics, such as the Westervelt or Kuznetsov equation, are by now pretty well-established; see, for example, Refs. [6–10] and the review paper.[11] Analysis of a local compressible Navier–Stokes system governing nonlinear sound motion can be found in Ref. [12]; see also Ref. [13] and the references contained therein.

Notation

Below we occasionally use $x \lesssim y$ for $x \leq Cy$, where $C > 0$ is a generic constant that does not depend on the Galerkin discretization parameter. We use subscript t to denote the temporal domain $(0, t)$ in Bochner spaces, where t is taken from a certain time interval to be specified; for example, $\|\cdot\|_{L^p_t(L^q(\Omega))}$ denotes the norm on $L^p(0, t; L^q(\Omega))$. If the subscript is omitted, the temporal domain is meant to be $(0, T)$.

2 | EXISTENCE OF SOLUTIONS

In this section, we provide the proof of existence of solutions of (5) with boundary and initial data given in (3) and (4), respectively. We first set the notion of the solution, where equations (mo$^\mu$) and (pd) will be understood in a time-integrated sense. More precisely, the solution space for the velocity is

$$X^\mu_{\boldsymbol{u}} = \left\{ \boldsymbol{u} \in L^\infty(0, T; H(\mathrm{div}; \Omega)) : \ \sqrt{\mu}\|\nabla(\nabla \cdot \boldsymbol{u})\|_{L^2(L^2(\Omega))} < \infty, \quad \boldsymbol{u} \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

endowed with the norm

$$\|\boldsymbol{u}\|_{X^\mu_{\boldsymbol{u}}} = \left\{ \|\nabla \cdot \boldsymbol{u}\|^2_{L^\infty(L^2(\Omega))} + \mu\|\nabla(\nabla \cdot \boldsymbol{u})\|^2_{L^2(L^2(\Omega))} \right\}^{1/2}.$$

Further, the solution space for the relative density is

$$X_\sigma = \left\{ \sigma \in H^1(0,T;H^{\frac{y}{2}}(\Omega)) \cap L^\infty(0,T;H^{\frac{y+1}{2}}(\Omega)) \,:\, \nabla\sigma \cdot \nu = 0 \text{ on } \partial\Omega \right\},\ y > d-1,\ 2 \le y \le 3, \tag{9}$$

with the norm

$$\|\sigma\|_{X_\sigma} = \left\{ \|\sigma\|^2_{L^\infty(H^{\frac{y+1}{2}}(\Omega))} + \|\sigma_t\|_{L^2(H^{\frac{y}{2}}(\Omega))} \right\}^{1/2}.$$

The assumptions made on y will be justified in the course of deriving energy estimates; see the discussion at the beginning of Section 2.3. We note that the condition $y \le 3$ can be removed if $g \equiv 0$. The setting $g = h \equiv 0$ is considered in Section 3.

Third, as we will prove existence of the time-integrated pressure $\mathrm{I}_t p$, we introduce the corresponding solution space as

$$X_{\mathrm{I}_t p} = \left\{ \mathrm{I}_t p = \int_0^t p(s)\,\mathrm{d}s \in L^2(0,T;H^1(\Omega)) \,:\, \nabla p \cdot \nu = 0 \text{ on } \partial\Omega,\quad \frac{1}{|\Omega|}\int_\Omega p\,\mathrm{d}x = 0 \right\}. \tag{10}$$

The targeted solution space for the studied problem is then $\mathcal{X}^\mu = X^\mu_{\boldsymbol{u}} \times X_\sigma \times X_{\mathrm{I}_t p}$.

Assumptions on data. We assume that the source term satisfies

$$\boldsymbol{f} \in X_{\boldsymbol{f}} = L^1(0,T;H_0(\mathrm{div};\Omega)) \cap L^2(0,T;L^2(\Omega)), \tag{11}$$

where $H_0(\mathrm{div};\Omega) = \{\boldsymbol{v} \in L^2(\Omega) \,:\, \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega,\ \nu\cdot\boldsymbol{v} = 0 \text{ on } \partial\Omega\}$. The initial conditions are assumed to satisfy

$$(\boldsymbol{u}_0, \boldsymbol{d}_0, \sigma_0) \in H(\mathrm{div};\Omega) \times \left(L^\infty(\Omega) \cap H^1(\Omega)\right) \times H^{\frac{y+1}{2}}(\Omega).$$

Additionally, we assume that

$$B/A \in X_{B/A} = L^\infty(\Omega) \cap W^{1,3}(\Omega) \tag{12}$$

and

$$\rho_0 \in X_{\rho_0} = \left\{ v \in L^\infty(\Omega) \,:\, \frac{1}{v} \in L^\infty(\Omega),\ \nabla \ln v \in L^\infty(\Omega) \cap H^{\frac{y+1}{2}}(\Omega) \right\} \tag{13}$$

as well as that

$$c_0^2 \in X_{c_0} = \left\{ v \in L^\infty(\Omega) \cap W^{1,3}(\Omega) \,:\, \frac{1}{v} \in L^\infty(\Omega) \right\}. \tag{14}$$

We next make precise what is meant by a solution of the problem.

Definition 1. We call $(\boldsymbol{u}, \sigma, p) \in \mathcal{X}^{\mu}$ a solution of problem (5) supplemented with boundary (3) and initial conditions (4) if it satisfies

$$\int_0^T \int_{\Omega} \Big\{ (\rho_0(\boldsymbol{u} - \boldsymbol{u}_0) + \nabla \mathrm{I}_t p - \mu \nabla (\nabla \cdot \boldsymbol{u}) - \mathrm{I}_t \boldsymbol{f}) \cdot \boldsymbol{v} + (\sigma_t + a(\sigma) \nabla \cdot \boldsymbol{u} - g(\boldsymbol{u})) v$$

$$+ \big(\mathrm{I}_t p - c_0^2 \rho_0 \mathrm{I}_t(b(\sigma)\sigma) - \mathrm{I}_t h(\boldsymbol{u}) \big) \Delta_{1/\rho_0} \phi + 2\alpha_0 \left(\tau (-\Delta)^{\frac{\gamma}{4}} (\sigma - \sigma_0) (-\Delta)^{\frac{\gamma}{4}} \phi + \eta (-\Delta)^{\frac{\gamma+1}{4}} \mathrm{I}_t \sigma (-\Delta)^{\frac{\gamma+1}{4}} \phi \right) \Big\} \, \mathrm{d}x \mathrm{d}t = 0$$

for all $\boldsymbol{v} \in L^2(0, T; L^2(\Omega)^d)$, $v \in L^2(0, T; L^2(\Omega))$, and $\phi \in L^2(0, T; H^{\frac{\gamma+1}{2}}(\Omega))$, such that $\nabla \phi \cdot \nu = 0$, with $\sigma_{|t=0} = \sigma_0$.

The proof of existence of solutions is set up through a Faedo–Galerkin procedure. To this end, we first need to construct suitable approximations of $(\boldsymbol{u}, \sigma, p)$.

2.1 | Construction of Galerkin approximations

We approximate the system in (5) by constructing a Galerkin approximation of (σ, p) by means of smooth eigenfunctions of the Neumann–Laplacian and then using it to set up suitable approximations of $\boldsymbol{u}$. This approach is in the spirit of Galerkin strategies for models of viscous compressible fluids; see Refs. [14–16] and the references provided therein. However, here the relative density σ and acoustic pressure p are directly approximated by means of suitable basis functions as opposed to the velocity $\boldsymbol{u}$.

Let $\{w_i\}_{i \geq 1}$ be the eigenfunctions of the Neumann–Laplacian operator $-\Delta_{1/\rho_0}$ acting on functions with zero mean, with eigenvalues $\{\lambda_i\}_{i \geq 1}$; that is, let

$$\begin{cases} -\Delta_{1/\rho_0} w_i = \lambda_i w_i & \text{in } \Omega, \\ \dfrac{1}{|\Omega|} \displaystyle\int_{\Omega} w_i \, \mathrm{d}x = 0, & \\ \nabla w_i \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix $n \in \mathbb{N}$ and let $W^n = \operatorname{span}\{w_1, \ldots, w_n\}$. We seek approximate σ and p in the form of

$$\sigma^n = \sum_{i=1}^{n} \xi_i^{\sigma,n}(t) w_i(x), \quad p^n = \sum_{i=1}^{n} \xi_i^{p,n}(t) w_i(x),$$

with the unknown time-dependent coefficients $\xi_i^{\sigma,n}$, $\xi_i^{p,n} : [0, T] \to \mathbb{R}$ for $i \in [1, n]$. Let the approximate initial relative density σ_0^n be the $H^{\frac{\gamma+1}{2}}(\Omega)$ projection of σ_0 on W^n. Denote $\boldsymbol{\xi}^n = [\xi_1^n \ \ldots \ \xi_n^n]^T$ and $\boldsymbol{\xi}_0^n = \boldsymbol{\xi}^n(0)$.

We then set $\boldsymbol{u}^n$ as the solution of the following system:

$$\begin{cases} (\mathrm{mo^G}) & \rho_0 \boldsymbol{u}_t^n - \mu \nabla (\nabla \cdot \boldsymbol{u}^n) + \nabla p^n = \boldsymbol{f} & \text{in } \Omega \times (0, T), & \boldsymbol{u}^n(0) = \boldsymbol{u}_0^n, \quad \nabla \boldsymbol{u}^n \cdot \nu = 0, \\ (\mathrm{ma^G}) & \sigma_t^n + a(\sigma^n) \nabla \cdot \boldsymbol{u}^n - \mathrm{g}(\boldsymbol{u}^n) = 0 & \text{in } W^n \times (0, T), & \sigma^n(0) = \sigma_0^n, \\ (\mathrm{pd^G}) & p^n = c_0^2 \rho_0 b(\sigma^n) \sigma^n - L\sigma^n + h(\boldsymbol{u}^n) & \text{in } W^n \times (0, T), & \end{cases} \tag{15}$$

where $g(\boldsymbol{u}^n) = -\boldsymbol{u}^n \cdot \nabla \ln \rho_0$ and $h(\boldsymbol{u}^n) = c_0^2(\mathrm{I}_t \boldsymbol{u}^n + \boldsymbol{d}_0) \cdot \nabla \rho_0$; cf. (8). By considering (ma$^{\mathrm{G}}$) and (pd$^{\mathrm{G}}$) in W^n, we mean that we project them onto the finite-dimensional space W^n with respect to the $L^2(\Omega)$ inner product. For showing that this approximation of (5) is well-posed, we need the following auxiliary existence result.

Lemma 1. *Let $\mu > 0$, $\rho_0, \frac{1}{\rho_0} \in L^\infty(\Omega)$, and $\boldsymbol{f} \in L^2(0,T;L^2(\Omega)^d)$. Let $\boldsymbol{u}_0^n \in H(div;\Omega)$. Then, given $p^n \in L^2(0,T;W^n)$, there exists a unique $\boldsymbol{u}^n \in X_{\boldsymbol{u}}^\mu \cap H^1(0,T;L^2(\Omega)^d)$ that satisfies*

$$\begin{cases} \boldsymbol{u}_t^n - \mu \dfrac{1}{\rho_0} \nabla(\nabla \cdot \boldsymbol{u}^n) = \dfrac{1}{\rho_0}(\boldsymbol{f} - \nabla p^n), \\ \boldsymbol{u}^n(0) = \boldsymbol{u}_0^n, \quad \boldsymbol{u}^n \cdot \nu = 0. \end{cases}$$

Proof. We observe that the right-hand side satisfies $\frac{1}{\rho_0}(\boldsymbol{f} - \nabla p^n) \in L^2(0,T;L^2(\Omega))$. The statement then follows along the lines of, for example, Ref. [17, Theorem 9.6]; we omit the details here. □

Lemma 1 allows us to define the solution operator $S : L^2(0,T;W^n) \to X_{\boldsymbol{u}}^\mu$, such that $S(p^n) = \boldsymbol{u}^n$. Let $p^{n,(1)}, p^{n,(2)} \in L^2(0,T;W^n)$, and denote $\boldsymbol{u}^{n,(1)} = S(p^{n,(1)})$ and $\boldsymbol{u}^{n,(2)} = S(p^{n,(2)})$. By testing the problem solved by $\boldsymbol{u}^{n,(1)} - \boldsymbol{u}^{n,(2)}$ with $-\nabla(\nabla \cdot (\boldsymbol{u}^{n,(1)} - \boldsymbol{u}^{n,(2)}))$, we conclude that this operator is globally Lipschitz continuous:

$$\begin{aligned} \|S(p^{n,(1)}) - S(p^{n,(2)})\|_{X_{\boldsymbol{u}}^\mu} &= \|\nabla \cdot (\boldsymbol{u}^{n,(1)} - \boldsymbol{u}^{n,(2)})\|_{L^\infty(L^2(\Omega))} + \sqrt{\mu}\|\nabla(\nabla \cdot (\boldsymbol{u}^{n,(1)} - \boldsymbol{u}^{n,(2)}))\|_{L^2(L^2(\Omega))} \\ &\leq C_0 \|\nabla p^{n,(1)} - \nabla p^{n,(2)}\|_{L^2(L^2(\Omega))} \\ &\leq C(n) \|p^{n,(1)} - p^{n,(2)}\|_{L^2(W^n)}, \end{aligned} \tag{16}$$

where the last line follows by the equivalence of norms in finite-dimensional spaces. The Galerkin problem then reduces to looking for a solution of

$$\begin{cases} \sigma^n = -a(\sigma^n)\nabla \cdot S(p^n) + g(S(p^n)) \quad \text{in } W^n \times (0,T), \\ \sigma^n(0) = \sigma_0^n, \\ p^n = c_0^2 \rho_0 b(\sigma^n)\sigma^n - L\sigma^n + h(S(p^n)) \quad \text{in } W^n \times (0,T), \end{cases} \tag{17}$$

which we tackle in the next step. The solution is at first obtained on an n-dependent interval $[0, T_n]$.

Proposition 1. *Let the assumptions of Lemma 1 hold with $\rho_0 \in X_{\rho_0}$, $B/A \in X_{B/A}$, and $c_0^2 \in X_{c_0}$. Then there exists $T_n = T_n(n) \in (0,T)$, such that problem (17) has a unique solution $(\sigma^n, p^n) \in H^1(0,T_n;W^n) \cap L^2(0,T_n;W^n)$.*

Proof. Let $R_1, R_2 > 0$. To prove unique solvability of (17), we apply Banach's fixed-point theorem on the mapping

$$\mathcal{T} : (\sigma_*^n, p_*^n) \mapsto (\sigma^n, p^n),$$

where (σ_*^n, p_*^n) is taken from the ball

$$B = \Big\{(\sigma_*^n, p_*^n) \in H^1(0,T;W^n) \times L^2(0,T;W^n) : \|\sigma_*^n\|_{H^1(0,T;L^2(\Omega))} \leq R_1,\ \|p_*^n\|_{L^2(L^2(\Omega))} \leq R_2, \\ \sigma_*^n(0) = \sigma_0^n \Big\},$$

and (σ^n, p^n) solves the following linear problem:

$$\begin{cases} \sigma_t^n = -a(\sigma_*^n)\nabla \cdot S(p_*^n) + g(S(p_*^n)) & \text{in } W^n \times (0,T), \\ \sigma^n(0) = \sigma_0^n, \\ p^n + L\sigma^n = c_0^2\rho_0 b(\sigma_*^n)\sigma_*^n + h(S(p_*^n)) & \text{in } W^n \times (0,T). \end{cases} \tag{18}$$

Self-mapping: Take $(\sigma_*^n, p_*^n) \in B$. We first check that $(\sigma^n, p^n) = \mathcal{T}(\sigma_*^n, p_*^n) \in B$. Note that

$$\begin{aligned} \|\sigma_t^n\|_{H^1(L^2(\Omega))} &\leq \|\sigma_t^n\|_{L^2(L^2(\Omega))} + \|\mathrm{I}_t\sigma_t^n + \sigma_0^n\|_{L^2(L^2(\Omega))} \\ &\leq (1+T)\|\sigma_t^n\|_{L^2(L^2(\Omega))} + \sqrt{T}\|\sigma_0^n\|_{L^2(\Omega)}. \end{aligned}$$

Using the first equation in (18), we then have

$$\begin{aligned} \|\sigma^n\|_{H^1(L^2(\Omega))} &\leq (1+T)\big(\|a(\sigma_*^n)\nabla \cdot S(p_*^n)\|_{L^2(L^2(\Omega))} + \|g(S(p_*^n))\|_{L^2(L^2(\Omega))}\big) + \sqrt{T}\|\sigma_0^n\|_{L^2(\Omega)} \\ &\leq (1+T)\sqrt{T}\big(\|a(\sigma_*^n)\|_{L^\infty(\Omega)}\|\nabla \cdot S(p_*^n)\|_{L^\infty(L^2(\Omega))} + \|g(S(p_*^n))\|_{L^\infty(L^2(\Omega))}\big) + \sqrt{T}\|\sigma_0^n\|_{L^2(\Omega)}. \end{aligned} \tag{19}$$

By relying on the estimate

$$\|g(S(p_*^n))\|_{L^2(L^2(\Omega))} = \|-S(p_*^n)\cdot \nabla \ln \rho_0\|_{L^2(L^2(\Omega))} \leq \|\nabla \ln \rho_0\|_{L^\infty(\Omega)}\sqrt{T}\|S(p_*^n)\|_{L^\infty(L^2(\Omega))}$$

and the equivalence of norms in finite-dimensional spaces, from (19), we conclude that

$$\|\sigma^n\|_{H^1(L^2(\Omega))} \leq C(n)(1+T)\sqrt{T}\big((1+R_1)R_2 + \|\nabla \ln \rho_0\|_{L^\infty(\Omega)}R_2\big) + \sqrt{T}\|\sigma_0^n\|_{L^2(\Omega)}.$$

We can thus guarantee that $\|\sigma^n\|_{H^1(L^2(\Omega))} \leq R_1$ by reducing $T = T(n)$.

From the last equation in (18) and the fact that $\|L(\sigma^n)\|_{L^2(L^2(\Omega))} \leq C(n)\|\sigma^n\|_{H^1(L^2(\Omega))}$, we can estimate p^n as follows:

$$\begin{aligned} \|p^n\|_{L^2(L^2(\Omega))} &\leq \sqrt{T}\|c_0^2\rho_0 b(\sigma_*^n)\sigma^n\|_{L^\infty(L^2(\Omega))} + \|L\sigma^n\|_{L^2(L^2(\Omega))} + \sqrt{T}\|h(S(p_*^n))\|_{L^\infty(L^2(\Omega))} \\ &\leq C(n)\Big(\sqrt{T}\|c_0^2\rho_0\|_{L^\infty(\Omega)}(1+\tfrac{1}{2}\|B/A\|_{L^\infty(\Omega)})R_1 + \|\sigma^n\|_{H^1(L^2(\Omega))} + \sqrt{T}\|c_0^2\nabla\rho_0\|_{L^\infty(\Omega)}R_2\Big) \\ &\quad + \sqrt{T}\|c_0^2\nabla\rho_0\|_{L^\infty(\Omega)}\|\boldsymbol{d}_0\|_{L^2(\Omega)}, \end{aligned} \tag{20}$$

where we have used the fact that

$$\|h(S(p_*^n))\|_{L^\infty(L^2(\Omega))} = \|c_0^2(\mathrm{I}_t(S(p_*^n)) + \boldsymbol{d}_0)\cdot\nabla\rho_0\|_{L^\infty(L^2(\Omega))}$$
$$\leq \|c_0^2\nabla\rho_0\|_{L^\infty(\Omega)}T\|S(p_*^n)\|_{L^\infty(L^2(\Omega))} + \|c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0\|_{L^2(\Omega)}.$$

Since we can reduce $\|\sigma^n\|_{H^1(L^2(\Omega))}$ by reducing the final time, from (20), we conclude that

$$\|p^n\|_{L^\infty(L^2(\Omega))} \leq R_2,$$

provided $T = T(n)$ is small enough.

Contractivity: Let $(\sigma_*^{n,(1)}, p_*^{n,(1)}), (\sigma_*^{n,(2)}, p_*^{n,(2)}) \in B$ and denote $(\sigma^{n,(1)}, p^{n,(1)}) = \mathcal{T}(\sigma_*^{n,(1)}, p_*^{n,(1)})$ and $(\sigma^{n,(2)}, p^{n,(2)}) = \mathcal{T}(\sigma_*^{n,(2)}, p_*^{n,(2)})$. Further, we introduce the following notation for the differences:

$$\overline{\sigma}_*^n = \sigma_*^{n,(1)} - \sigma_*^{n,(2)}, \qquad \overline{\sigma}^n = \sigma^{n,(1)} - \sigma^{n,(2)},$$
$$\overline{p}_*^n = p_*^{n,(1)} - p_*^{n,(2)}, \qquad \overline{p}^n = p^{n,(1)} - p^{n,(2)}.$$

We can see $(\overline{\sigma}^n, \overline{p}^n)$ as the solution to the following problem:

$$\begin{cases} \overline{\sigma}_t^n = -a(\sigma_*^{n,(1)})\nabla\cdot(S(p_*^{n,(1)}) - S(p_*^{n,(2)})) - 2\overline{\sigma}_*^n\nabla\cdot S(p_*^{n,(2)}) + g\Big(S(p_*^{n,(1)}) - S(p_*^{n,(2)})\Big) & \text{in } W^n\times(0,T), \\ \overline{\sigma}^n(0) = 0, & \\ \overline{p}^n + L\overline{\sigma}^n = c_0^2\rho_0\dfrac{B}{2A}\overline{\sigma}_*^n\sigma_*^{n,(1)} + c_0^2\rho_0 b(\sigma_*^{n,(2)})\overline{\sigma}_*^n + h\Big(S(p_*^{n,(1)}) - S(p_*^{n,(2)})\Big) & \text{in } W^n\times(0,T), \end{cases}$$

where we have used the fact that $a(\sigma_*^{n,(1)}) - a(\sigma_*^{n,(2)}) = 2\overline{\sigma}_*^n$ and $b(\sigma_*^{n,(1)}) - b(\sigma_*^{n,(2)}) = \frac{B}{2A}\overline{\sigma}_*^n$; cf. (7). Similarly to (19), we then have the following estimate:

$$\|\overline{\sigma}^n\|_{H^1(L^2(\Omega))} \leq (1+T)\sqrt{T}\Big(\|a(\sigma_*^{n,(1)})\nabla\cdot(S(p_*^{n,(1)}) - S(p_*^{n,(2)}))\|_{L^\infty(L^2(\Omega))} + 2\|\overline{\sigma}_*^n\nabla\cdot S(p_*^{n,(2)})\|_{L^\infty(L^2(\Omega))}$$
$$+ \|g\Big(S(p_*^{n,(1)}) - S(p_*^{n,(2)})\Big)\|_{L^\infty(L^2(\Omega))}.$$

By relying on the fact that

$$\left\|a\left(\sigma_*^{n,(1)}\right)\right\|_{L^\infty(L^\infty(\Omega))} \leq C(n)(1+R_1),$$
$$\left\|\nabla\cdot S\left(p_*^{n,(2)}\right)\right\|_{L^\infty(L^\infty(\Omega))} \leq C(n)\left\|\nabla\cdot S\left(p_*^{n,(2)}\right)\right\|_{L^\infty(L^2(\Omega))} \leq C(n)\|p_*^{n,(2)}\|_{L^2(L^2(\Omega))} \leq C(n)R_2,$$

together with the Lipschitz continuity of S (see (16)) and

$$\|g\Big(S(p_*^{n,(1)}) - S\left(p_*^{n,(2)}\right)\Big)\|_{L^\infty(L^2(\Omega))} \leq C(n)\|\nabla\ln\rho_0\|_{L^\infty(\Omega)}\|\overline{p}_*^n\|_{L^2(L^2(\Omega))},$$

we obtain

$$\|\overline{\sigma}^n\|_{H^1(L^2(\Omega))} \lesssim C(n)(1+T)\sqrt{T}\Big(\|\overline{\sigma}^n_*\|_{L^\infty(L^2(\Omega))} + \|\overline{p}^n_*\|_{L^2(L^2(\Omega))}\Big). \tag{21}$$

We can bound the differences of pressures as follows:

$$\begin{aligned}\|\overline{p}^n\|_{L^2(L^2(\Omega))} \leq{}& \sqrt{T}\|c_0^2\rho_0\tfrac{B}{2A}\overline{\sigma}^n_*\sigma^{n,(1)}_*\|_{L^\infty(L^2(\Omega))} + \sqrt{T}\|c_0^2\rho_0 b(\sigma^{n,(2)}_*)\overline{\sigma}^n_*\|_{L^\infty(L^2(\Omega))} + \|L(\overline{\sigma}^n)\|_{L^2(L^2(\Omega))} \\ &+ \|h(S(p^{n,(1)}_*) - S(p^{n,(2)}_*))\|_{L^2(L^2(\Omega))}.\end{aligned} \tag{22}$$

By the equivalence of norms in finite-dimensional spaces and estimate (21), we infer

$$\|L(\overline{\sigma}^n)\|_{L^2(L^2(\Omega))} \leq C(n)\|\overline{\sigma}^n\|_{H^1(L^2(\Omega))} \lesssim C(n)\sqrt{T}\Big(\|\overline{\sigma}^n_*\|_{L^\infty(L^2(\Omega))} + \|\overline{p}^n_*\|_{L^2(L^2(\Omega))}\Big).$$

Further,

$$\begin{aligned}\|h(S(p^{n,(1)}_*) - S(p^{n,(2)}_*))\|_{L^2(L^2(\Omega))} &\leq \sqrt{T}\|c_0^2 \mathrm{I}_t\Big(S(p^{n,(1)}_*) - S(p^{n,(2)}_*)\Big)\cdot\nabla\rho_0\|_{L^\infty(L^2(\Omega))} \\ &\leq \sqrt{T}\|c_0^2\nabla\rho_0\|_{L^\infty(\Omega)}\|S(p^{n,(1)}_*) - S(p^{n,(2)}_*\|_{L^\infty(L^2(\Omega))} \\ &\leq C(n)\sqrt{T}\|\overline{p}^n_*\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Using this bound in (22) together with the Lipschitz continuity of the operator S yields

$$\|\overline{p}^n\|_{L^2(L^2(\Omega))} \leq C(n)\sqrt{T}\Big(\|\overline{\sigma}^n_*\|_{H^1(L^2(\Omega))} + \|\overline{p}^n_*\|_{L^2(L^2(\Omega))}\Big). \tag{23}$$

By adding the two bounds, (21) and (23), we arrive at

$$\|\overline{\sigma}^n\|_{H^1(L^2(\Omega))} + \|\overline{p}^n\|_{L^2(L^2(\Omega))} \leq C(n)(1+T)\sqrt{T}\Big(\|\overline{\sigma}^n_*\|_{L^\infty(L^2(\Omega))} + \|\overline{p}^n_*\|_{L^2(L^2(\Omega))}\Big).$$

Thus, strict contractivity of the mapping can be guaranteed by reducing $T = T(n)$. An application of Banach's fixed-point theorem yields the statement. □

2.2 | Energy identity for Galerkin approximations

Having constructed Galerkin approximations, in the next step, we derive an energy identity for (15) on $[0, T_n]$. For this purpose, we introduce $\mathrm{P}^{\rho_0}_{W^n}\mathfrak{g} = \mathrm{P}^{\rho_0}_{W^n}[\mathrm{g}(\boldsymbol{u}^n)] \in W^n$ as the Ritz projection of $\mathfrak{g} = \mathrm{g}(\boldsymbol{u}^n) = -\boldsymbol{u}^n \cdot \nabla \ln \rho_0$ in the sense of

$$\int_\Omega \frac{1}{\rho_0}\nabla \mathrm{g}(\boldsymbol{u}^n)\cdot\nabla v_n \,\mathrm{d}x = \int_\Omega \frac{1}{\rho_0}\nabla \mathrm{P}^{\rho_0}_{W^n}\mathfrak{g}\cdot\nabla v_n\,\mathrm{d}x \qquad \text{for all } v_n \in W^n; \tag{24}$$

that is,

$$(-\Delta_{1/\rho_0}\mathrm{g}(\boldsymbol{u}^n),\, v_n)_{L^2} = (-\Delta_{1/\rho_0}\mathrm{P}^{\rho_0}_{W^n}\mathfrak{g},\, v_n)_{L^2} \qquad \text{for all } v_n \in W^n.$$

In the derivation of the energy identity for $(\boldsymbol{u}^n, \sigma^n, p^n)$, we rely on the stability of this projection operator in the following sense.

Lemma 2. *For* $\mathfrak{g} = g(\boldsymbol{u}^n) = -\nabla \ln \rho_0 \cdot \boldsymbol{u}^n$, *where* $\boldsymbol{u}^n \in L^\infty(\Omega) \cap H^{\frac{y+1}{2}}(\Omega)$, $\rho_0 \in X_{\rho_0}$, $\|\nabla \ln \rho_0\|_{L^\infty(\Omega)\cap H^{\frac{y+1}{2}}(\Omega)} \leq 1$, *we have*

$$
\begin{aligned}
&\|\nabla \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \leq C \|\nabla \ln \rho_0\|_{L^\infty(\Omega)\cap W^{1,3}(\Omega)} \|\boldsymbol{u}^n\|_{H^1(\Omega)},\\
&\|(-\Delta_N)^{\frac{y}{4}} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \leq C \|\nabla \ln \rho_0\|_{L^\infty(\Omega)\cap H^{\frac{y}{2}}(\Omega)} \|\boldsymbol{u}^n\|_{L^\infty(\Omega)\cap H^{\frac{y}{2}}(\Omega)},\\
&\|(-\Delta_N)^{\frac{y+1}{4}} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \leq C \|\nabla \ln \rho_0\|_{L^\infty(\Omega)\cap H^{\frac{y+1}{2}}(\Omega)} \|\boldsymbol{u}^n\|_{L^\infty(\Omega)\cap H^{\frac{y+1}{2}}(\Omega)},
\end{aligned}
\tag{25}
$$

with C depending only on $\|\rho_0\|_{L^\infty(\Omega)}$, $\|\frac{1}{\rho_0}\|_{L^\infty(\Omega)}$, *but not on n.*

Proof. The proof is provided in the Appendix. □

We proceed to derive an energy identity for $(\boldsymbol{u}^n, \sigma^n, p^n)$ on $[0, T_n]$ under the assumption of uniform smallness of solutions on $[0, T_n]$.

Proposition 2. *Let the assumptions of Lemma 1 and Proposition 1 hold with* $\boldsymbol{f} \in X_{\boldsymbol{f}}$. *Let* $(\boldsymbol{u}^n, \sigma^n, p^n)$ *be the solution of (15) on* $[0, T_n]$. *Assume that there exists* $r > 0$, *independent of n, such that*

$$
|\sigma^n(x,t)| \leq r \quad \text{for all } (x,t) \in W^n \times [0, T_n]. \tag{26}
$$

Then if $r > 0$ *is sufficiently small, there exist* $\underline{a}$, $\overline{a} > 0$ *and* $\underline{b}$, $\overline{b} > 0$, *independent of n, such that*

$$
\begin{aligned}
&0 < \underline{a} \leq a(\sigma^n) \leq \overline{a} \qquad \text{for all } (x,t) \in W^n \times [0, T_n],\\
&0 < \underline{b} \leq b(\sigma^n) \leq \overline{b} \qquad \text{for all } (x,t) \in W^n \times [0, T_n],
\end{aligned}
\tag{27}
$$

and the following identity holds:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\sqrt{a(\sigma^n)}\nabla \cdot \boldsymbol{u}^n\|^2_{L^2(\Omega)} + \|c_0\sqrt{b(\sigma^n)}\,\nabla\sigma^n\|^2_{L^2(\Omega)}\Big) + \mu \|\sqrt{(a(\sigma^n))/\rho_0}\nabla(\nabla \cdot \boldsymbol{u}^n)\|^2_{L^2(\Omega)}\\
&+ \alpha_0\Big(2\tau\|(-\Delta_N)^{\frac{y}{4}}\sigma^n_t\|^2_{L^2(\Omega)} + \eta\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta_N)^{\frac{y+1}{4}}\sigma^n\|^2_{L^2(\Omega)}\Big) = \mathrm{rhs}_1 + \mathrm{rhs}_2,
\end{aligned}
\tag{28}
$$

where the right-hand side terms are given by

$$
\begin{aligned}
\mathrm{rhs}_1 =& -\int_\Omega a(\sigma^n)\nabla \cdot (\rho_0^{-1}\boldsymbol{f})\nabla \cdot \boldsymbol{u}^n \,\mathrm{d}x + \int_\Omega \frac{1}{2}c_0^2 b'(\sigma^n)\sigma^n_t|\nabla\sigma^n|^2\,\mathrm{d}x\\
&- \int_\Omega \sigma^n\big(\nabla[c_0^2 b(\sigma^n)] + c_0^2 b(\sigma^n)\nabla \ln \rho_0\big)\cdot\nabla\sigma^n_t\,\mathrm{d}x + \frac{1}{2}\int_\Omega a'(\sigma^n)\sigma^n_t|\nabla\cdot\boldsymbol{u}^n|^2\\
&+ \mu\int_\Omega \frac{1}{\rho_0}\nabla(\nabla\cdot\boldsymbol{u}^n)\cdot(a'(\sigma^n)\nabla\sigma^n)\nabla\cdot\boldsymbol{u}^n\,\mathrm{d}x
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
\mathrm{rhs}_2 = &-\int_\Omega \frac{1}{\rho_0}\nabla h(\boldsymbol{u}^n)\cdot\nabla\sigma_t^n\,\mathrm{d}x\\
&+\int_\Omega\Big(\frac{1}{\rho_0}\nabla \mathrm{P}^{\rho_0}_{W^n}\mathrm{g}(\boldsymbol{u}^n)\cdot\nabla[c_0^2\rho_0 b(\sigma^n)\sigma^n - h(\boldsymbol{u}^n)] - 2\alpha_0\big(\tau(-\Delta_N)^{\frac{y}{2}}\sigma_t^n + \eta(-\Delta_N)^{\frac{y+1}{2}}\sigma^n\big)\mathrm{P}^{\rho_0}_{W^n}\mathrm{g}(\boldsymbol{u}^n)\Big)\,\mathrm{d}x
\end{aligned}
\tag{30}
$$

with $a'(\sigma^n) = 2$ *and* $b'(\sigma^n) = \frac{B}{2A}$.

Proof. Since $a(\sigma^n) = 1 + 2\sigma^n$ and $b(\sigma^n) = 1 + \frac{B}{2A}\sigma^n$, the bounds in (27) follow immediately by (26) if r is small enough. The identity in (28) is obtained by convenient testing of the problem that will lead to cancellations of several terms. We test equation (mo^{G}) in (15) with

$$
\boldsymbol{v}^n = -\frac{1}{\rho_0}\nabla(a(\sigma^n(t))\nabla\cdot\boldsymbol{u}^n(t)),
$$

equation (ma^{G}) with $-\Delta_{1/\rho_0}p^n(t)$, and equation (pd^{G}) with $\Delta_{1/\rho_0}\sigma_t^n(t)$. We note that we are allowed to do this because $\boldsymbol{v}^n \in L^2(\Omega)^d$ and $-\Delta_{1/\rho_0}p^n(t)$, $\Delta_{1/\rho_0}\sigma_t^n(t) \in W^n$. Proceeding in this manner, integrating over Ω, and integrating by parts in space yields

$$
\begin{aligned}
&\int_\Omega -(\rho_0\boldsymbol{u}_t^n + \nabla p^n - \mu\nabla(\nabla\cdot\boldsymbol{u}^n) - \boldsymbol{f})\cdot\frac{1}{\rho_0}\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\,\mathrm{d}x\\
&+\int_\Omega \nabla\big(\sigma_t^n + a(\sigma^n)\nabla\cdot\boldsymbol{u}^n - \mathrm{g}(\boldsymbol{u}^n)\big)\cdot\frac{1}{\rho_0}\nabla p^n\,\mathrm{d}x\\
&-\int_\Omega \nabla\big(p^n - c_0^2\rho_0 b(\sigma^n)\sigma^n - h(\boldsymbol{u}^n)\big)\cdot\frac{1}{\rho_0}\nabla\sigma_t^n\,\mathrm{d}x\\
&+2\alpha_0\int_\Omega(-\Delta_{1/\rho_0})^{-1}\Big(\tau(-\Delta)^{\frac{y}{2}}\sigma_t^n + \eta(-\Delta)^{\frac{y+1}{2}}\sigma^n\Big)(-\Delta_{1/\rho_0}\sigma_t^n)\,\mathrm{d}x = 0
\end{aligned}
$$

a.e. in time. Conveniently, the (space-integrated) terms $-\nabla p^n\cdot\frac{1}{\rho_0}\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)$ and $\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\cdot\frac{1}{\rho_0}\nabla p^n$ as well as $\nabla\sigma_t^n\cdot\frac{1}{\rho_0}\nabla p^n$ and $-\nabla p^n\cdot\frac{1}{\rho_0}\nabla\sigma_t^n$ cancel out and we are left with

$$
\begin{aligned}
&\int_\Omega -(\rho_0\boldsymbol{u}_t^n - \mu\nabla(\nabla\cdot\boldsymbol{u}^n) - \boldsymbol{f})\cdot\frac{1}{\rho_0}\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\,\mathrm{d}x - \int_\Omega\nabla\mathrm{g}(\boldsymbol{u}^n)\cdot\frac{1}{\rho_0}\nabla p^n\,\mathrm{d}x\\
&-\int_\Omega\nabla\big(-c_0^2\rho_0 b(\sigma^n)\sigma^n - h(\boldsymbol{u}^n)\big)\cdot\frac{1}{\rho_0}\nabla\sigma_t^n\,\mathrm{d}x + 2\alpha_0\int_\Omega(\tau(-\Delta)^{\frac{y}{2}}\sigma_t^n + \eta(-\Delta)^{\frac{y+1}{2}}\sigma^n)\sigma_t^n\,\mathrm{d}x = 0.
\end{aligned}
$$

To transform the terms further, we can employ the following identities:

$$
\begin{aligned}
-\int_\Omega \boldsymbol{u}_t^n\cdot\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\,\mathrm{d}x &= \int_\Omega a(\sigma^n)\nabla\cdot\boldsymbol{u}_t^n\nabla\cdot\boldsymbol{u}^n\,\mathrm{d}x - \int_{\partial\Omega}(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\boldsymbol{u}_t^n\cdot\nu\,\mathrm{d}S\\
&= \int_\Omega\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{a(\sigma^n)}\nabla\cdot\boldsymbol{u}^n|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega a'(\sigma^n)\sigma_t^n|\nabla\cdot\boldsymbol{u}^n|^2\,\mathrm{d}x
\end{aligned}
$$

and

$$\mu \int_\Omega \nabla(\nabla\cdot\boldsymbol{u}^n)\cdot\frac{1}{\rho_0}\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\,\mathrm{d}x = \mu\|\sqrt{a(\sigma^n)/\rho_0}\nabla(\nabla\cdot\boldsymbol{u}^n)\|^2_{L^2(\Omega)} + \mu\int_\Omega \nabla(\nabla\cdot\boldsymbol{u}^n)\cdot\frac{1}{\rho_0}\nabla a(\sigma^n)\nabla\cdot\boldsymbol{u}^n\,\mathrm{d}x,$$

as well as, with $\beta = c_0^2 b(\sigma^n)$,

$$\nabla[\rho_0\beta\sigma^n]\cdot\frac{1}{\rho_0}\nabla\sigma_t^n = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\beta}\nabla\sigma^n|^2 - \frac{1}{2}\beta_t|\nabla\sigma^n|^2 + \sigma^n(\nabla\beta + \beta\nabla\ln\rho_0)\cdot\nabla\sigma_t^n,$$

where $\beta_t = c_0^2\frac{B}{2A}\sigma_t^n$. In this way, we obtain the energy identity

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\sqrt{a(\sigma^n)}\nabla\cdot\boldsymbol{u}^n\|^2_{L^2(\Omega)} + \|c_0\sqrt{b(\sigma^n)}\,\nabla\sigma^n\|^2_{L^2(\Omega)}\Big) + \mu\|\sqrt{a(\sigma^n)/\rho_0}\nabla(\nabla\cdot\boldsymbol{u}^n)\|^2_{L^2(\Omega)}\\
&+\alpha_0\Big(2\tau\|(-\Delta)^{\frac{y}{4}}\sigma_t^n\|^2_{L^2(\Omega)} + \eta\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{\frac{y+1}{4}}\sigma^n\|^2_{L^2(\Omega)}\Big)\\
&= -\int_\Omega \frac{1}{\rho_0}\boldsymbol{f}\cdot\nabla(a(\sigma^n)\nabla\cdot\boldsymbol{u}^n)\,\mathrm{d}x + \int_\Omega\frac{1}{\rho_0}\nabla g(\boldsymbol{u}^n)\cdot\nabla p^n\,\mathrm{d}x - \int_\Omega\frac{1}{\rho_0}\nabla h(\boldsymbol{u}^n)\cdot\nabla\sigma_t^n\,\mathrm{d}x \qquad (31)\\
&+\int_\Omega\frac{1}{2}c_0^2 b'(\sigma^n)\sigma_t^n|\nabla\sigma^n|^2\,\mathrm{d}x - \int_\Omega \sigma^n(\nabla[c_0^2 b(\sigma^n)] + c_0^2 b(\sigma^n)\nabla\ln\rho_0)\cdot\nabla\sigma_t^n\,\mathrm{d}x\\
&+\frac{1}{2}\int_\Omega a'(\sigma^n)\sigma_t^n|\nabla\cdot\boldsymbol{u}^n|^2\,\mathrm{d}x + \mu\int_\Omega\frac{1}{\rho_0}\nabla(\nabla\cdot\boldsymbol{u}^n)\cdot\nabla a(\sigma^n)\nabla\cdot\boldsymbol{u}^n\,\mathrm{d}x := \mathrm{rhs}.
\end{aligned}$$

Note that the term with ∇p^n on the right-hand side of (31) cannot be controlled directly by the left-hand side terms so we would not be able to derive an energy estimate starting from (31). To mend this, we rewrite this term by additionally testing equation (pd^{G}) in (15) with $-\Delta_{1/\rho_0}\mathrm{P}^{\rho_0}_{W^n} g(\boldsymbol{u}^n) \in W^n$. We then have

$$\begin{aligned}
\int_\Omega \frac{1}{\rho_0}\nabla g(\boldsymbol{u}^n)\cdot\nabla p^n\,\mathrm{d}x &= \int_\Omega\frac{1}{\rho_0}\nabla\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)\cdot\nabla p^n\,\mathrm{d}x\\
&= \int_\Omega\Big(\frac{1}{\rho_0}\nabla\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)\cdot\nabla[c_0^2\rho_0 b(\sigma^n)\sigma^n - h(\boldsymbol{u}^n)]\\
&\quad - 2\alpha_0\big(\tau(-\Delta)^{\frac{y}{2}}\sigma_t^n + \eta(-\Delta)^{\frac{y+1}{2}}\sigma^n\big)\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)\Big)\,\mathrm{d}x.
\end{aligned}$$

Using this identity, the right-hand side of (31) can be rewritten as the sum $\mathrm{rhs} = \mathrm{rhs}_1 + \mathrm{rhs}_2$, where rhs_1 is defined in (29) and rhs_2 in (30), to arrive at the claim. □

2.3 | Energy estimate

Starting from the obtained identity in (28), we next derive an energy estimate, at first, on $[0, T_n]$ and again under an assumption of uniform smallness of solutions. Concerning the regularity induced by the y-power damping terms on the left-hand side of (31), there are several requirements that we needed to take into account:

- First of all, we need to obtain a bound on σ from the η term in (31) whose control in its turn enables nondegeneracy of $a(\sigma^n) = 1 + 2\sigma^n$ and $b(\sigma^n) = 1 + \frac{B}{2A}\sigma^n$. Thus, we require that $2\frac{y+1}{4} > \frac{d}{2}$.
- Second, rhs_1, given in (29), contains the gradient of σ_t, which we have to control by the left-hand side term $2\alpha_0\tau\|(-\Delta_N)^{\frac{y}{4}}\sigma_t^n\|^2_{L^2(\Omega)}$ in (31), resulting in the requirement $2\frac{y}{4} \geq 1$.
- Third, to be able to absorb the $g(\boldsymbol{u}^n)$ terms in rhs_2, defined in (30), by the left-hand side, we need an upper bound on y: $y \leq 3$.

Altogether, we thus assume that the propagation medium exhibits attenuation with the exponent

$$y > d - 1 \text{and} 2 \leq y \leq 3, \quad d \in \{2, 3\}. \tag{32}$$

As mentioned before, the condition $y \leq 3$ can be removed if $g \equiv 0$. The case $g = h \equiv 0$ is analyzed in Section 3 in a μ-uniform manner for which the lower bound on y has to be strengthened, however.

In the analysis below, we use the Poincaré–Friedrichs inequality as well as elliptic regularity of the Neumann problem (Ref. [18, Theorem 4, p. 217]) to conclude existence of constants C_s, $\tilde{C}_s$, such that

$$\|\phi\|_{H^s(\Omega)} \leq C_s\|(-\Delta_N)^{s/2}\phi\|_{L^2(\Omega)} \leq \tilde{C}_s\|\phi\|_{H^s(\Omega)} \text{ for all } \phi \in H^s(\Omega), \int_\Omega \phi = 0, \quad s \in \left\{\frac{y}{2}, \frac{y+1}{2}\right\}.$$

We note that, under the assumptions (32) made on y, we have continuity of the embeddings

$$H^{\frac{y}{2}}(\Omega) \to H^1(\Omega) \to L^6(\Omega), \quad H^{\frac{y+1}{2}}(\Omega) \to L^\infty(\Omega) \cap W^{1,3}(\Omega) \tag{33}$$

for $d \in \{2, 3\}$. We next derive a uniform bound for the sum of the semidiscrete energy and dissipation functionals at time t given by

$$\mathcal{E}(t) = \|\boldsymbol{u}^n(t)\|^2_{H(\text{div};\Omega)} + \|\nabla\sigma^n(t)\|^2_{L^2(\Omega)} + \|\sigma^n(t)\|^2_{H^{\frac{y+1}{2}}(\Omega)}$$

and

$$\mathcal{D}(t) = \int_0^t \left(\mu\|\nabla(\nabla \cdot \boldsymbol{u}^n(s))\|^2_{L^2(\Omega)} + \|\sigma_t^n(s)\|^2_{H^{\frac{y}{2}}(\Omega)} + \|\nabla \text{I}_s p\|^2_{L^2(\Omega)}\right) \text{d}s,$$

at first, for $t \in [0, T_n]$. In the subsequent step, we will use this result to bootstrap the existence and the energy bounds to $[0, T]$.

Proposition 3. *Let the assumptions of Proposition 2 hold and let the approximate initial velocity* $\boldsymbol{u}_0^n \in H(div;\Omega)$ *satisfy*

$$\boldsymbol{u}_0^n \to \boldsymbol{u}_0 \text{ in } H(div;\Omega) \quad \text{as } n \to \infty.$$

Let the condition (32) on y as well as

$$\|\nabla[c_0^2\nabla\rho_0]\|_{L^2(\Omega)} + \|c_0^2\nabla\rho_0\|_{L^3(\Omega)} + \|\nabla\ln\rho_0\|_{H^{\frac{y+1}{2}}(\Omega)} < \delta_{\rho_0,c_0} \tag{34}$$

hold. Furthermore, assume that there exist $r > 0$, independent of n, such that

$$\begin{aligned}\|\sigma^n\|_{L^\infty(0,T_n;L^\infty(\Omega))} + \|a'(\sigma^n)\sigma_t^n\|_{L^2(0,T_n;L^3(\Omega))} + \|a'(\sigma^n)\nabla\sigma^n\|_{L^\infty(0,T_n;L^3(\Omega))} + \|c_0^2b'(\sigma^n)\sigma_t^n\|_{L^2(0,T_n;L^6(\Omega))}\\ +\|\nabla[c_0^2b(\sigma^n)]\|_{L^\infty(0,T_n;L^2(\Omega))} + \|c_0^2b(\sigma^n)\nabla\ln\rho_0\|_{L^\infty(0,T_n;L^2(\Omega))} < r\end{aligned} \tag{35}$$

with $a'(\sigma^n) = 2$ and $b'(\sigma^n) = \frac{B}{2A}$. Then for sufficiently small r and sufficiently small δ_{ρ_0,c_0}, independently of n, the following bound holds:

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in(0,T_n)}\mathcal{E}(t) + \operatorname*{ess\,sup}_{t\in(0,T_n)}\mathcal{D}(t)\\ &\leq C_1(\rho_0)\exp(C_2T_n)\Big(\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^2(L^2(\Omega))} + \|\mathrm{I}_t\boldsymbol{f}\|^2_{L^2(L^2(\Omega))} + \|\boldsymbol{u}_0\|^2_{H(div;\Omega)} + \|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{H^{\frac{y+1}{2}}(\Omega)}\\ &\quad + \|\nabla[c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0]\|^2_{L^2(\Omega)}\Big),\end{aligned} \tag{36}$$

where C_1 and C_2 do not depend on T_n or n.

Note that the smallness assumption on the gradients of c_0 and ρ_0 made in (34) only restricts their variations but still allows for large absolute values of these quantities.

Proof. We start from the derived energy identity in (31) and estimate the right-hand side terms within rhs_1 and rhs_2.

Estimate of rhs_1: The time integral of the first right-hand side term rhs_1 can be bounded using Hölder's inequality and the fact that $a'(\sigma^n) = 2$ as follows:

$$\begin{aligned}\int_0^t \text{rhs}_1(s)\,\mathrm{d}s \leq\; &\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|_{L^1(L^2(\Omega))}\overline{a}\|\nabla\cdot\boldsymbol{u}^n\|_{L_t^\infty(L^2(\Omega))} + \frac{1}{2}\|c_0^2b'(\sigma^n)\sigma_t^n\|_{L_t^2(L^6(\Omega))}\|\nabla\sigma^n\|_{L_t^2(L^3(\Omega))}\|\nabla\sigma^n\|_{L_t^\infty(L^2(\Omega))}\\ &+ \|\sigma^n\|_{L_t^2(L^\infty(\Omega))}\|\nabla\sigma_t^n\|_{L_t^2(L^2(\Omega))}\Big(\|\nabla[c_0^2b(\sigma^n)]\|_{L_t^\infty(L^2(\Omega))} + \|c_0^2b(\sigma^n)\nabla\ln\rho_0\|_{L_t^\infty(L^2(\Omega))}\Big)\\ &+ \|\sigma_t^n\|_{L_t^2(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L_t^\infty(L^2(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L_t^2(L^6(\Omega))}\\ &+ 2\mu\|1/\rho_0\|_{L^\infty(\Omega)}\|\nabla\sigma^n\|_{L_t^\infty(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L_t^2(L^6(\Omega))}\|\nabla(\nabla\cdot\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}\end{aligned} \tag{37}$$

for $t\in[0,T_n]$. By employing the assumed r bound and Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\|c_0^2b'(\sigma^n)\sigma_t^n\|_{L_t^2(L^6(\Omega))}\|\nabla\sigma^n\|_{L_t^2(L^3(\Omega))}\|\nabla\sigma^n\|_{L_t^\infty(L^2(\Omega))} &\leq \frac{1}{2}r\cdot\|\nabla\sigma^n\|_{L_t^2(L^3(\Omega))}\|\nabla\sigma^n\|_{L_t^\infty(L^2(\Omega))}\\ &\leq r^2\|\sigma^n\|^2_{L_t^\infty(H^{\frac{y+1}{2}}(\Omega))} + \frac{1}{4}C(\Omega)\|\sigma^n\|^2_{L_t^2(H^{\frac{y+1}{2}}(\Omega))}\end{aligned}$$

since $\|c_0^2 b'(\sigma^n)\sigma_t^n\|_{L^2(0,T_n;L^6(\Omega))} \leq r$. Similarly,

$$
\begin{aligned}
&\|\sigma^n\|_{L^2_t(L^\infty(\Omega))}\|\nabla\sigma_t^n\|_{L^2_t(L^2(\Omega))}\Big(\|\nabla[c_0^2 b(\sigma^n)]\|_{L^\infty_t(L^2(\Omega))} + \|c_0^2 b(\sigma^n)\nabla \ln \rho_0\|_{L^\infty_t(L^2(\Omega))}\Big)\\
&\leq \|\sigma^n\|_{L^2_t(L^\infty(\Omega))}\|\nabla\sigma_t^n\|_{L^2_t(L^2(\Omega))}\cdot r\\
&\leq \frac{1}{4\varepsilon}C(\Omega)\|\sigma^n\|^2_{L^2_t(H^{\frac{\gamma+1}{2}}(\Omega))} + \varepsilon r^2\|\sigma_t^n\|^2_{L^2_t(H^{\frac{\gamma}{2}}(\Omega))}
\end{aligned}
$$

for any $\varepsilon > 0$. Note that by the first embedding in (33), we have

$$
\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^6(\Omega))} \lesssim \|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(H^1(\Omega))} \lesssim \|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^2(\Omega))} + \|\nabla(\nabla\cdot\boldsymbol{u}^n)\|_{L^2_t(L^2(\Omega))}.
$$

Thus, we can estimate the last two terms in (37) using also the assumed r bound as follows:

$$
\begin{aligned}
&\|\sigma_t^n\|_{L^2_t(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^\infty_t(L^2(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^6(\Omega))}\\
&\quad + 2\mu\|1/\rho_0\|_{L^\infty(\Omega)}\|\nabla\sigma^n\|_{L^\infty_t(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^6(\Omega))}\|\nabla(\nabla\cdot\boldsymbol{u}^n)\|_{L^2_t(L^2(\Omega))}\\
&\lesssim r\|\nabla\cdot\boldsymbol{u}^n\|_{L^\infty_t(L^2(\Omega))}(\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^2(\Omega))} + \|\nabla\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^2(\Omega))})\\
&\quad + \mu\|1/\rho_0\|_{L^\infty(\Omega)} r(\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^2(\Omega))} + \|\nabla\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^2(\Omega))})\|\nabla(\nabla\cdot\boldsymbol{u}^n)\|_{L^2_t(L^2(\Omega))}.
\end{aligned}
$$

Then by applying Young's inequality, we obtain

$$
\begin{aligned}
&\|\sigma_t^n\|_{L^2_t(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^\infty_t(L^2(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^6(\Omega))}\\
&\quad + 2\mu\|1/\rho_0\|_{L^\infty(\Omega)}\|\nabla\sigma^n\|_{L^\infty_t(L^3(\Omega))}\|\nabla\cdot\boldsymbol{u}^n\|_{L^2_t(L^6(\Omega))}\|\nabla(\nabla\cdot\boldsymbol{u}^n)\|_{L^2_t(L^2(\Omega))}\\
&\lesssim r^2\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^\infty_t(L^2(\Omega))} + (\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))} + \varepsilon\|\nabla\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))}) + \mu r\|\nabla\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))}\\
&\quad + \|1/\rho_0\|^2_{L^\infty(\Omega)}\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))} + \mu^2 r^2\|\nabla\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))}
\end{aligned}
$$

for any $\varepsilon > 0$. Note that the presence of the ε term above will lead to a nonuniform estimate in μ. In Section 3, we discuss a way to mend this by requiring more regularity from σ_t^n. By employing the derived bounds in (33), we arrive at

$$
\begin{aligned}
\int_0^t \mathrm{rhs}_1(s)\,\mathrm{d}s \lesssim{}& \frac{1}{4\varepsilon}\overline{a}^{-2}\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^1(L^2(\Omega))} + \varepsilon\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^\infty_t(L^2(\Omega))}\\
&+ \|\sigma^n\|^2_{L^2_t(H^{\frac{\gamma+1}{2}}(\Omega))} + r^2\|\sigma^n\|_{L^\infty_t(H^{\frac{\gamma+1}{2}}(\Omega))} + \varepsilon r^2\|\sigma_t^n\|_{L^2_t(H^{\frac{\gamma}{2}}(\Omega))} + r^2\|\nabla\cdot\boldsymbol{u}^n\|_{L^\infty_t(L^2(\Omega))}\\
&+ (1 + \|1/\rho_0\|^2_{L^\infty(\Omega)})\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2_t(L^2(\Omega))} + (\mu r(1+\mu r) + \varepsilon)\|\nabla(\nabla\cdot\boldsymbol{u}^n)\|^2_{L^2_t(L^2(\Omega))})
\end{aligned} \tag{38}
$$

for any $\varepsilon > 0$, where the hidden constant does not depend on n. The σ^n and $\boldsymbol{u}^n$ terms on the right-hand side above will be either absorbed for small enough ε and r or tackled via Grönwall's inequality in the final stages of the proof.

Estimate of rhs_2: Next, we estimate the time integral of rhs_2, given in (30), by employing Hölder's inequality as follows:

$$\int_0^t \text{rhs}_2(s)\,\mathrm{d}s \le \|1/\rho_0\|_{L^\infty(\Omega)}\Big\{\Big(\|\nabla h(\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}\|\nabla\sigma_t^n\|_{L_t^2(L^2(\Omega))}$$
$$+\|\nabla \mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}\|\nabla[c_0^2\rho_0 b(\sigma^n)\,\sigma^n]\|_{L_t^2(L^2(\Omega))}+\|\nabla \mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}\|\nabla h(\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}\Big)$$
$$+2\alpha_0\Big(\tau\|(-\Delta)^{\frac{y}{4}}\sigma_t^n\|_{L_t^2(L^2(\Omega))}\|(-\Delta)^{\frac{y}{4}}\mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|_{L_t^2(L^2(\Omega))}$$
$$+\eta\|(-\Delta)^{\frac{y+1}{4}}\sigma^n\|_{L_t^\infty(L^2(\Omega))}\|(-\Delta)^{\frac{y+1}{4}}\mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|_{L_t^1(L^2(\Omega))}\Big)\Big\}.$$

By employing Young's inequality, we obtain

$$\int_0^t \text{rhs}_2(s)\,\mathrm{d}s \le \|1/\rho_0\|_{L^\infty(\Omega)}\Big\{\Big(\frac{1}{4\varepsilon}\|\nabla h(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))}+\varepsilon\|\nabla\sigma_t^n\|^2_{L_t^2(L^2(\Omega))}$$
$$+\varepsilon\|\nabla g(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))}+\frac{1}{4\varepsilon}\|\nabla[c_0^2\rho_0 b(\sigma^n)\,\sigma^n]\|^2_{L_t^2(L^2(\Omega))}+\varepsilon\|\nabla g(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))}$$
$$+\frac{1}{4\varepsilon}\|\nabla h(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))}\Big)+2\alpha_0\Big(\varepsilon\tau^2\|(-\Delta)^{\frac{y}{4}}\sigma_t^n\|^2_{L_t^2(L^2(\Omega))}+\frac{1}{4\varepsilon}\|(-\Delta)^{\frac{y}{4}}\mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))}$$
$$+\eta^2\varepsilon\|(-\Delta)^{\frac{y+1}{4}}\sigma^n\|^2_{L_t^\infty(L^2(\Omega))}+\frac{1}{4\varepsilon}\|(-\Delta)^{\frac{y+1}{4}}\mathrm{P}_{W^n}^{\rho_0} g(\boldsymbol{u}^n)\|^2_{L_t^1(L^2(\Omega))}\Big)\Big\} \tag{39}$$

for any $\varepsilon > 0$. We further have

$$\varepsilon\|\nabla g(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))} = \varepsilon\|\nabla(\boldsymbol{u}^n\cdot\nabla\ln\rho_0)\|^2_{L_t^2(L^2(\Omega))}$$
$$\le 2\varepsilon\|\nabla\boldsymbol{u}^n\|^2_{L_t^2(L^6(\Omega))}\|\nabla\ln\rho_0\|^2_{L^3(\Omega)}+2\varepsilon\|\boldsymbol{u}^n\|^2_{L_t^2(L^\infty(\Omega))}\|\nabla(\nabla\ln\rho_0)\|^2_{L^2(\Omega)}$$

and

$$\frac{1}{4\varepsilon}\|\nabla h(\boldsymbol{u}^n)\|^2_{L_t^2(L^2(\Omega))} \le \frac{3}{4\varepsilon}\|\nabla[c_0^2\nabla\rho_0]\|^2_{L^2(\Omega)}\|\mathrm{I}_s\boldsymbol{u}^n\|^2_{L_t^2(L^\infty(\Omega))}+\frac{3}{4\varepsilon}\|c_0^2\nabla\rho_0\|^2_{L^3(\Omega)}\|\mathrm{I}_s\nabla\boldsymbol{u}^n\|^2_{L_t^2(L^6(\Omega))}$$
$$+\frac{3}{4\varepsilon}\|\nabla[c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0]\|^2_{L_t^2(L^2(\Omega))}$$

and the arising $\boldsymbol{u}^n$ terms on the right-hand side can be absorbed by $\mu\int_0^t\|\nabla(\nabla\cdot\boldsymbol{u}^n(s))\|^2_{L^2(\Omega)}\,\mathrm{d}s$ for sufficiently small $\varepsilon > 0$ and δ_{ρ_0,c_0}. Furthermore,

$$\frac{1}{4\varepsilon}\|\nabla[c_0^2\rho_0 b(\sigma^n)\,\sigma^n]\|^2_{L_t^2(L^2(\Omega))}$$
$$\le\frac{1}{2\varepsilon}\|\nabla[c_0^2\rho_0]b(\sigma^n)+c_0^2\rho_0\nabla\frac{B}{2A}\sigma^n+c_0^2\rho_0\frac{B}{2A}\nabla\sigma^n\|^2_{L_t^\infty(L^2(\Omega))}\|\sigma^n\|^2_{L_t^2(L^\infty(\Omega))}+\frac{1}{2\varepsilon}\|c_0^2\rho_0 b(\sigma^n)\|^2_{L_t^\infty(L^\infty(\Omega))}\|\nabla\sigma^n\|^2_{L_t^2(L^2(\Omega))}.$$

From here,

$$\frac{1}{4\varepsilon}\|\nabla[c_0^2\rho_0 b(\sigma^n)\,\sigma^n]\|^2_{L^2_t(L^2(\Omega))}$$
$$\lesssim \Big(\|\nabla[c_0^2\rho_0]\|^2_{L^\infty(\Omega)}(1+\|\tfrac{B}{2A}\|_{L^\infty(\Omega)}r)^2 + \|c_0^2\rho_0\nabla\tfrac{B}{2A}\|^2_{L^3(\Omega)}r^2 + \|c_0^2\rho_0\tfrac{B}{2A}\|^2_{L^\infty(\Omega)}r^2\Big)\|\sigma^n\|^2_{L^2_t(L^\infty(\Omega))}$$
$$+\|c_0^2\rho_0\|^2_{L^\infty(\Omega)}(1+r)^2\|\nabla\sigma^n\|^2_{L^2_t(L^2(\Omega))}$$

and these terms can be handled via Grönwall's inequality.

We can use the stability of $\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)$ according to Lemma 2 to estimate

$$\frac{1}{4\varepsilon}\|(-\Delta)^{\frac{\gamma}{4}}\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)\|^2_{L^2_t(L^2(\Omega))} + \frac{1}{4\varepsilon}\|(-\Delta)^{\frac{\gamma+1}{4}}\mathrm{P}^{\rho_0}_{W^n}g(\boldsymbol{u}^n)\|^2_{L^1_t(L^2(\Omega))}$$
$$\leq C\|\nabla\ln\rho_0\|_{L^\infty(\Omega)\cap H^{\frac{\gamma}{2}}(\Omega)}\|\boldsymbol{u}^n\|_{L^2_t(L^\infty(\Omega)\cap H^{\frac{\gamma}{2}}(\Omega))} + C\sqrt{T}\|\nabla\ln\rho_0\|_{L^\infty(\Omega)\cap H^{\frac{\gamma+1}{2}}(\Omega)}\|\boldsymbol{u}^n\|_{L^2_t(L^\infty(\Omega)\cap H^{\frac{\gamma+1}{2}}(\Omega))},$$

where C does not depend on n, and absorb these terms for sufficiently small $\|\nabla\ln\rho_0\|_{L^\infty(\Omega)\cap H^{\frac{\gamma+1}{2}}(\Omega)}$ (i.e., δ_{ρ_0,c_0}).

Combining the bounds: By employing (38) and (39) in the time-integrated identity (28), taking the supremum over $t\in(0,\tau)$ for $\tau\in(0,T_n)$ and reducing ε and r (independently of n), we end up with

$$\operatorname*{ess\,sup}_{t\in(0,\tau)}\mathcal{E}(t) + \int_0^\tau\Big(\mu\|\nabla(\nabla\cdot\boldsymbol{u}^n(s))\|^2_{L^2(\Omega)} + \|(-\Delta_N)^{\frac{\gamma}{4}}\sigma^n_t(s)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}s$$
$$\leq C\Big(\|\boldsymbol{u}_0\|^2_{H(\mathrm{div};\Omega)} + \|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{H^{\frac{\gamma+1}{2}}(\Omega)} + \frac{1}{4\varepsilon}\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^1(L^2(\Omega))}$$
$$+\|\sigma^n\|^2_{L^2(0,\tau;H^{\frac{\gamma+1}{2}}(\Omega))} + (1+\|1/\rho_0\|^2_{L^\infty(\Omega)})\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2(0,\tau;L^2(\Omega))} + T_n\|\nabla[c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0]\|^2_{L^2(\Omega)}\Big) \tag{40}$$

for $\tau\in(0,T_n)$, where C does not depend on T_n or n. Above, we have also used the boundedness of approximate initial data:

$$\|\nabla\cdot\boldsymbol{u}_0^n\|_{L^2(\Omega)}\lesssim\|\boldsymbol{u}_0\|_{H(\mathrm{div};\Omega)},\qquad \|\sigma_0^n\|_{H^{\frac{\gamma+1}{2}}(\Omega)}\lesssim\|\sigma_0\|_{H^{\frac{\gamma+1}{2}}(\Omega)}.$$

Estimate (41) does not contain a bound on p^n, which we obtain in the final step of the proof. To this end, we test the time-integrated version of (mo$^{\mathrm{G}}$) with $\nabla(\mathrm{I}_t p^n)\in L^2(\Omega)^d$ for $t\in[0,T_n]$, which yields, after integration over Ω,

$$\int_\Omega\Big\{\big(\rho_0(\boldsymbol{u}^n-\boldsymbol{u}^n_0) + \nabla\mathrm{I}_t p^n - \mu\nabla(\nabla\cdot\mathrm{I}_t\boldsymbol{u}) - \mathrm{I}_t\boldsymbol{f}\big)\cdot\nabla(\mathrm{I}_t p^n)\,\mathrm{d}x = 0.$$

From here, we obtain

$$\|\nabla\mathrm{I}_t p^n\|_{L^2(0,\tau;L^2(\Omega))}\leq\|\mathrm{I}_t\boldsymbol{f}+\mu\nabla(\nabla\cdot\mathrm{I}_t\boldsymbol{u}^n)-\rho_0(\boldsymbol{u}^n-\boldsymbol{u}^n_0)\|_{L^2(0,\tau;L^2(\Omega))}$$
$$\lesssim\|\mathrm{I}_t\boldsymbol{f}\|_{L^2(L^2(\Omega))}+\mu\|\nabla(\nabla\cdot\mathrm{I}_t\boldsymbol{u}^n)\|_{L^2(0,\tau;L^2(\Omega))}+\|\rho_0\|_{L^\infty(\Omega)}\|\boldsymbol{u}^n\|_{L^2(0,\tau;L^2(\Omega))}+\|\rho_0\|_{L^\infty(\Omega)}\|\boldsymbol{u}^n_0\|_{L^2(\Omega)}.$$

Note that we cannot obtain a bound on ∇p^n from (mo$^{\mathrm{G}}$) because we lack a bound on $\boldsymbol{u}_t^n$. Squaring this estimate, multiplying it by $\lambda > 0$, and adding it to (41) with λ sufficiently small (independently of n) leads to

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,\tau)} \mathcal{E}(t) + \operatorname*{ess\,sup}_{t\in(0,\tau)} \mathcal{D}(t) \\
&\leq C\Big((1+\|\rho_0\|^2_{L^\infty(\Omega)})\|\boldsymbol{u}_0\|^2_{H(\mathrm{div};\Omega)} + \|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{H^{\frac{\gamma+1}{2}}(\Omega)} + \|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^1(L^2(\Omega))} + \|\mathrm{I}_t\boldsymbol{f}\|^2_{L^2(L^2(\Omega))} \\
&\quad + \|\rho_0\|^2_{L^\infty(\Omega)}\|\boldsymbol{u}^n\|^2_{L^2(0,\tau;L^2(\Omega))} + \|\sigma^n\|^2_{L^2(0,\tau;H^{\frac{\gamma+1}{2}}(\Omega))} + (1+\|1/\rho_0\|_{L^\infty(\Omega)})\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2(0,\tau;L^2(\Omega))} \\
&\quad + T_n\|\nabla[c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0]\|^2_{L^2(\Omega)}\Big)
\end{aligned}
\tag{41}
$$

for $t \in [0, T_n]$, where the constant C does not depend on n. By employing Grönwall's inequality, we arrive at the claimed estimate. □

2.4 | Extending the existence interval to $[0, T]$

Equipped with a uniform bound in (36), we can now extend the existence interval of Galerkin approximations to $[0, T]$. We do so by proving that for small enough data, the uniform boundedness assumption made in (35) holds.

Proposition 4. *Let the assumptions of Lemma 1 and Proposition 1 hold. Let assumption (32) on γ as well as assumption (34) on the smallness of gradients of ρ_0 and c_0^2 hold with the bound δ_{ρ_0,c_0}. Further, let $(\boldsymbol{u}^n, \sigma^n, p^n)$ be the solution of (15) on $[0, T_n]$. Then there exists $\delta > 0$, independent of n, such that if*

$$
\|\sigma_0\|^2_{H^{\frac{\gamma+1}{2}}(\Omega)} + \|\boldsymbol{u}_0\|^2_{H(div;\Omega)} + \|\boldsymbol{d}_0\|^2_{L^\infty(\Omega)\cap H^1(\Omega)} + \|\boldsymbol{f}\|^2_{X_{\boldsymbol{f}}} \leq \delta, \tag{42}
$$

and δ_{ρ_0,c_0} is small enough, independent of n, then the following uniform bound holds:

$$
\begin{aligned}
\mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t) := \|\sigma^n(t)\|_{L^\infty(\Omega)} + 2\|\sigma_t^n\|_{L^2(0,t;L^3(\Omega))} + 2\|\nabla\sigma^n(t)\|_{L^3(\Omega)} + \frac{1}{2}\|(B/A)c_0^2\sigma_t^n\|_{L^2(0,t;L^6(\Omega))} \\
+\|\nabla[c_0^2 b(\sigma^n)](t)\|_{L^2(\Omega)} + \|c_0^2 b(\sigma^n)(t)\nabla\ln\rho_0\|_{L^2(\Omega)} < r
\end{aligned}
$$

for all $t \in [0, T_n]$. Consequently, $T_n = T$ can be chosen independent of n.

Proof. We argue by contradiction. Assume that there exists $t_0 \in [0, T_n]$, such that

$$
\mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t_0) > r.
$$

Let $t_* = \inf\{t : \mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t) > r\}$. By continuity of $\mathcal{L}$, then

$$
\mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t_*) = r.
$$

However, since $\mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t_*) = r$, we know from the energy bound (36) that

$$\mathcal{E}(t_*) + \mathcal{D}(t_*) \leq C\delta. \tag{43}$$

Furthermore, by employing the Sobolev embeddings in (33), it follows that

$$\mathcal{L}^2(\sigma^n, \boldsymbol{u}^n)(t_0) \leq C_0(\mathcal{E}(t_*) + \mathcal{D}(t_*)), \tag{44}$$

where (crucially) the constant C_0 does not depend on T_n or n. Combining estimates (43) and (44) yields

$$\mathcal{L}^2(\sigma^n, \boldsymbol{u}^n)(t_*) \leq CC_0\delta.$$

Choosing the size of data to be $\delta < \frac{r^2}{CC_0}$ leads to $\mathcal{L}(\sigma^n, \boldsymbol{u}^n)(t_*) < r$ and thus a contradiction.

By Proposition 3, the uniform boundedness of $\mathcal{L}$ in turn implies that the energy is uniformly bounded:

$$\mathcal{E}(t) \leq C, \quad t \in [0, T_n],$$

and we can thus prolong Galerkin solutions until we reach the final time T. □

2.5 | Passing to the limit as $n \to \infty$

Thanks to the established n-uniform bounds on Galerkin approximations on $[0, T]$, we may extract subsequences of $\{\boldsymbol{u}^n\}_{n\geq 1}$ and $\{\sigma^n\}_{n\geq 1}$, which we do not relabel, such that

$$\begin{array}{llll} \boldsymbol{u}^n & \rightharpoonup \boldsymbol{u} & \text{weakly-}* & \text{in} \quad L^\infty(0, T; H(\text{div}; \Omega)), \\ \nabla \cdot \boldsymbol{u}^n & \rightharpoonup \nabla \cdot \boldsymbol{u} & \text{weakly-}* & \text{in} \quad L^\infty(0, T; L^2(\Omega)), \\ \nabla(\nabla \cdot \boldsymbol{u}^n) & \rightharpoonup \nabla(\nabla \cdot \boldsymbol{u}) & \text{weakly} & \text{in} \quad L^2(0, T; L^2(\Omega)), \end{array} \tag{45}$$

and

$$\begin{array}{llll} \sigma^n & \rightharpoonup \sigma & \text{weakly-}* & \text{in} \quad L^\infty(0, T; H^{\frac{\gamma+1}{2}}(\Omega)), \\ \sigma^n_t & \rightharpoonup \sigma_t & \text{weakly} & \text{in} \quad L^2(0, T; H^{\frac{\gamma}{2}}(\Omega)). \end{array} \tag{46}$$

By the compact embedding $X_\sigma \hookrightarrow\hookrightarrow C([0, T]; H^1(\Omega))$, we also know that there is a subsequence of $\{\sigma^n\}_{n\geq 1}$, not relabeled, such that

$$\sigma^n \to \sigma \quad \text{strongly in } C([0, T]; H^1(\Omega)). \tag{47}$$

Additionally, by the uniform boundedness of $\mathrm{I}_t p^n$, there is a subsequence of $\{\mathrm{I}_t p^n)\}_{n\geq 1}$, again not relabeled, such that

$$I_t p^n \rightharpoonup I_t p \text{ weakly in } L^2(0, T; H^1(\Omega)). \tag{48}$$

Thanks to (45) and (48), we can immediately pass to the limit as $n \to \infty$ in

$$\int_0^T \int_\Omega \big(\rho_0(\boldsymbol{u}^n - \boldsymbol{u}_0^n) + \nabla \mathrm{I}_t p^n - \mu \nabla(\nabla \cdot \mathrm{I}_t \boldsymbol{u}^n) - \mathrm{I}_t \boldsymbol{f}\big) \cdot \boldsymbol{v} \,\mathrm{d}x\mathrm{d}t = 0, \quad \boldsymbol{v} \in L^2(0,T;L^2(\Omega)^d)$$

to conclude that

$$\int_0^T \int_\Omega (\rho_0(\boldsymbol{u} - \boldsymbol{u}_0) + \nabla \mathrm{I}_t p - \mu \nabla(\nabla \cdot \mathrm{I}_t \boldsymbol{u}) - \mathrm{I}_t \boldsymbol{f}) \cdot \boldsymbol{v} \,\mathrm{d}x\mathrm{d}t = 0 \quad \text{for all } \boldsymbol{v} \in L^2(0,T;L^2(\Omega)^d).$$

Next, to pass to the limit in (ma^{G}), we first note that for any $w \in L^2(0,T;L^2(\Omega))$, we have

$$\int_0^T \int_\Omega (\sigma^n \nabla \cdot \boldsymbol{u}^n - \sigma \nabla \cdot \boldsymbol{u}) w \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega (\sigma^n - \sigma) \nabla \cdot \boldsymbol{u}^n w \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega (\nabla \cdot \boldsymbol{u}^n - \nabla \cdot \boldsymbol{u}) \sigma w \,\mathrm{d}x\mathrm{d}t. \tag{49}$$

The convergence of the second term to zero as $n \to \infty$ is immediate since $\sigma w \in L^2(0,T;L^2(\Omega))$ for $w \in L^2(0,T;L^2(\Omega))$. The convergence of the first term to zero follows by

$$\int_0^T \int_\Omega (\sigma^n - \sigma) \nabla \cdot \boldsymbol{u}^n w \,\mathrm{d}x\mathrm{d}t \le C \|\sigma^n - \sigma\|_{C([0,T];H^1(\Omega))} \|\nabla \cdot \boldsymbol{u}^n\|_{L^2(0,T;L^4(\Omega))} \|w\|_{L^2(0,T;L^2(\Omega))}$$

and (47). Next, we fix N and choose

$$v(t) = \sum_{i=1}^N \xi_i^\sigma(t) w_i(x), \quad \phi(t) = \sum_{i=1}^N \xi_i^p(t) w_i(x), \tag{50}$$

where $\{\xi^\sigma\}_{i=1}^N$ and $\{\xi^p\}_{i=1}^N$ are given smooth functions. We choose $n \ge N$ and note that σ^n satisfies

$$\int_0^T \int_\Omega \big(\sigma_t^n + a(\sigma^n) \nabla \cdot \boldsymbol{u}^n - \mathrm{g}(\boldsymbol{u}^n)\big) v \,\mathrm{d}x\mathrm{d}t = 0.$$

Thanks to the convergence of (49) to zero as $n \to \infty$, we can then pass to the limit as $n \to \infty$ in the above equation and use the density of functions of the form (50) in $L^2(0,T;L^2(\Omega))$ to conclude that

$$\int_0^T \int_\Omega (\sigma_t + a(\sigma) \nabla \cdot \boldsymbol{u} - \mathrm{g}(\boldsymbol{u})) v \,\mathrm{d}x\mathrm{d}t = 0 \quad \text{for any } v \in L^2(0,T;L^2(\Omega)).$$

Similarly to the arguments in, for example, Ref. [19, Ch. 7], with $v(T) = 0$, we have

$$-\int_\Omega \sigma^n(0) v(0) \,\mathrm{d}x - \int_0^T \int_\Omega \sigma v_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega (a(\sigma^n) \nabla \cdot \boldsymbol{u}^n - \mathrm{g}(\boldsymbol{u}^n)) v \,\mathrm{d}x\mathrm{d}t = 0.$$

By passing to the limit as $n \to \infty$ and using the analogous identity for σ, we can show that $\sigma(0) = \sigma_0$ since $\sigma^n(0) \to \sigma_0$ in $L^2(\Omega)$. With similar reasoning to (49),

$$\int_0^T \int_\Omega (\mathrm{I}_t(b(\sigma^n)\sigma^n) - \mathrm{I}_t(b(\sigma)\sigma))\phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \mathrm{I}_t(\sigma^n - \sigma)\phi \,\mathrm{d}x\mathrm{d}t + \frac{B}{2A}\int_0^T \int_\Omega \mathrm{I}_t((\sigma^n - \sigma)(\sigma^n + \sigma))\phi \,\mathrm{d}x\mathrm{d}t \to 0$$

as $n \to \infty$, thanks to (46) and (47). We can then pass to the limit also in

$$\int_0^T \int_\Omega \Big\{ (\mathrm{I}_t p^n - c_0^2\rho_0 \mathrm{I}_t(b(\sigma^n)\sigma^n) - \mathrm{I}_t h(\boldsymbol{u}^n))\Delta_{1/\rho_0}\phi + 2\alpha_0\big(\tau(-\Delta)^{\frac{y}{4}}(\sigma^n - \sigma_0^n)(-\Delta)^{\frac{y}{4}}\phi + \eta(-\Delta)^{\frac{y+1}{4}}\mathrm{I}_t\sigma^n(-\Delta)^{\frac{y+1}{4}}\phi\big)\Big\}\,\mathrm{d}x = 0,$$

using that

$$((-\Delta)^{\frac{y}{4}}(\sigma_0 - \sigma_0^n), (-\Delta)^{\frac{y}{4}}\phi)_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Altogether, we conclude that $(\boldsymbol{u}, \sigma, p)$ is a solution of the problem in the sense of Definition 1.

By passing to the limit in the semidiscrete energy estimate (36) and utilizing the lower semicontinuity of norms, we find that $(\boldsymbol{u}, \sigma, p)$ satisfies an energy bound analogous to (36) and arrive at the following existence result.

Theorem 1 (Existence of solutions). *Let $T > 0$. Let $\mu > 0$ and $d - 1 < y$, $2 \le y \le 3$ (cf. (32)), and assume that*

$$\boldsymbol{u}_0 \in H(div;\Omega), \quad \boldsymbol{d}_0 \in L^\infty(\Omega) \cap H^1(\Omega), \quad \sigma_0 \in H^{\frac{y+1}{2}}(\Omega), \quad \boldsymbol{f} \in X_{\boldsymbol{f}},$$

and $B/A \in X_{B/A}$, $\rho_0 \in X_{\rho_0}$, $c_0^2 \in X_{c_0}$, where the spaces $X_{\boldsymbol{f}}$, $X_{B/A}$, X_{ρ_0}, and X_{c_0} are defined in (11), (12), (13), and (14), respectively. There exist $\delta > 0$ and $\delta_{\rho_0,c_0} > 0$, such that if the smallness conditions (34) and (42) hold, then there exists a solution $(\boldsymbol{u}, \sigma, p)$ of (5) in the sense of Definition 1, which satisfies the following bound:

$$\begin{aligned}&\|\boldsymbol{u}\|^2_{L^\infty(H(div;\Omega))} + \|\nabla\sigma\|^2_{L^\infty(L^2(\Omega))} + \|\sigma\|^2_{L^\infty(H^{\frac{y+1}{2}}(\Omega))} + \int_0^T \Big(\mu\|\nabla(\nabla\cdot\boldsymbol{u}(t))\|^2_{L^2(\Omega)} + \|\sigma_t(t)\|^2_{H^{\frac{y}{2}}(\Omega)} + \|\nabla \mathrm{I}_t p\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t \\ &\le C_1\exp(C_2 T)\Big(\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^2(L^2(\Omega))} + \|\mathrm{I}_t\boldsymbol{f}\|^2_{L^2(L^2(\Omega))} + \|\boldsymbol{u}_0\|^2_{H(div;\Omega)} + \|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{H^{\frac{y+1}{2}}(\Omega)} \\ &\quad + \|\nabla[c_0^2\boldsymbol{d}_0\cdot\nabla\rho_0]\|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{51}$$

The estimate in (51) is not uniform in μ; that is, we cannot use this result to investigate the limit of solutions as $\mu \searrow 0$. As the setting $\mu = 0$ is of interest for working with (1), we investigate it next by modifying the assumptions on y as well as the functions g and h.

3 | THE VANISHING VISCOSITY LIMIT UNDER STRONGER ASSUMPTIONS

In this section, we discuss the vanishing μ limit of solutions to the problem with $g = h \equiv 0$:

$$\int_0^T \int_\Omega \{(\rho_0(\boldsymbol{u}-\boldsymbol{u}_0) + \nabla I_t p - \mu\nabla(\nabla\cdot\boldsymbol{u}) - I_t \boldsymbol{f})\cdot\boldsymbol{v} + (\sigma_t + a(\sigma)\nabla\cdot\boldsymbol{u})\,v$$
$$+ \left(I_t p - c_0^2\rho_0 I_t(b(\sigma)\sigma)\right)\Delta_{1/\rho_0}\phi + 2\alpha_0\left(\tau(-\Delta)^{\frac{y}{4}}(\sigma-\sigma_0)(-\Delta)^{\frac{y}{4}}\phi + \eta(-\Delta)^{\frac{y+1}{4}} I_t\sigma(-\Delta)^{\frac{y+1}{4}}\phi\right)\Big\}\, dxdt = 0 \tag{52}$$

which holds for all $\boldsymbol{v} \in L^2(0,T;L^2(\Omega)^d)$, $v \in L^2(0,T;L^2(\Omega))$, and $\phi \in L^2(0,T;H^{\frac{y+1}{2}}(\Omega))$, such that $\nabla\phi\cdot\nu = 0$.

Looking at the energy estimates in the previous section starting from the identity in (28), we see that we can simplify them because now $\text{rhs}_2 \equiv 0$. Since $g = h = 0$, we can assume slightly less regularity of the coefficients as compared to (13), (14), namely that

$$\rho_0, \frac{1}{\rho_0}, c_0, \frac{1}{c_0}, B/A \in L^\infty(\Omega), \quad \rho_0 \in H^2(\Omega), \quad c_0^2, B/A \in W^{1,3}(\Omega). \tag{53}$$

Furthermore, there is no need for the initial condition on $\boldsymbol{d}$. By then re-examining the derivation of the estimate of the time-integrated rhs_1, we observe that the culprit in (29) for the nonuniform bounds in μ was the term

$$\int_0^t \sigma_t^n|\nabla\cdot\boldsymbol{u}^n|\,\mathrm{d}s \le \int_0^t \|\sigma_t^n\|_{L^6(\Omega)}\|\nabla\cdot\boldsymbol{u}^n\|_{L^2(\Omega)}\|\nabla\cdot\boldsymbol{u}^n\|_{L^3(\Omega)}\,\mathrm{d}s, \tag{54}$$

in particular, the need to further bound $\|\nabla\cdot\boldsymbol{u}^n\|_{L^3(\Omega)}$; see (37). We can obtain a μ-uniform energy estimate if we have the following bound:

$$\|\sigma_t^n\|_{L^\infty(\Omega)} \le C\|\sigma_t^n\|_{H^{\frac{y}{2}}(\Omega)}, \tag{55}$$

because we can then replace estimate (54) by

$$\int_0^t \sigma_t^n|\nabla\cdot\boldsymbol{u}^n|\,\mathrm{d}s \le \int_0^t \|\sigma_t^n\|_{L^\infty(\Omega)}\|\nabla\cdot\boldsymbol{u}^n\|^2_{L^2(\Omega)}\,\mathrm{d}s.$$

For this reason, here we strengthen the lower bound on y to $y > d$, so that embedding estimate (55) holds. (Note that since $g = h = 0$, we do not need the condition $y \le 3$ any longer). By otherwise proceeding as in the previous section via the Faedo–Galerkin procedure, we arrive at the following uniform in μ result.

Proposition 5. *Let $T > 0$. Let $\mu > 0$ and $y > d$ and assume that*

$$\boldsymbol{u}_0 \in H(div;\Omega), \quad \sigma_0 \in H^{\frac{y+1}{2}}(\Omega), \quad \boldsymbol{f} \in X_{\boldsymbol{f}},$$

and that (53) holds. There exists $\delta > 0$, such that if

$$\|\sigma_0\|^2_{H^{\frac{y+1}{2}}(\Omega)} + \|\boldsymbol{u}_0\|^2_{H(div;\Omega)} + \|\boldsymbol{f}\|^2_{X_{\boldsymbol{f}}} \le \delta,$$

then there exists a solution $(\boldsymbol{u},\sigma,p)$ *of (52), which satisfies the following bound:*

$$\|\boldsymbol{u}\|^2_{L^\infty(H(div;\Omega))}+\|\nabla\sigma\|^2_{L^\infty(L^2(\Omega))}+\|\sigma\|^2_{L^\infty(H^{\frac{y+1}{2}}(\Omega))}+\int_0^T\left(\mu\|\nabla(\nabla\cdot\boldsymbol{u}(t))\|^2_{L^2(\Omega)}+\|\sigma_t(t)\|^2_{H^{\frac{y}{2}}(\Omega)}+\|\nabla \mathrm{I}_t p\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$

$$\leq C_1\exp(C_2T)\Big(\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^2(L^2(\Omega))}+\|\mathrm{I}_t\boldsymbol{f}\|^2_{L^2(L^2(\Omega))}+\|\boldsymbol{u}_0\|^2_{H(div;\Omega)}+\|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)}+\|\sigma_0\|^2_{H^{\frac{y+1}{2}}(\Omega)}\Big),\tag{56}$$

where the constants C_1 *and* C_2 *do not depend on* μ*.*

As (58) provides a μ-uniform bound on $(\boldsymbol{u},\sigma,\mathrm{I}_t p)$, similarly to Section 2, we can find subsequences of $\{\boldsymbol{u}\}_{\mu>0}$ and $\{\sigma\}_{\mu>0}$, which we do not relabel, such that

$$\begin{aligned}\boldsymbol{u}&\rightharpoonup\boldsymbol{u}^{\mu=0}&&\text{weakly-}*\ \text{in}\ L^\infty(0,T;H(\mathrm{div};\Omega)),\\ \nabla\cdot\boldsymbol{u}&\rightharpoonup\nabla\cdot\boldsymbol{u}^{\mu=0}&&\text{weakly-}*\ \text{in}\ L^\infty(0,T;L^2(\Omega)),\end{aligned}$$

and

$$\begin{aligned}\sigma&\rightharpoonup\sigma^{\mu=0}&&\text{weakly-}*\ \text{in}\ L^\infty(0,T;H^{\frac{y+1}{2}}(\Omega)),\\ \sigma_t&\rightharpoonup\sigma_t^{\mu=0}&&\text{weakly}\ \text{in}\ L^2(0,T;H^{\frac{y}{2}}(\Omega))\end{aligned}$$

as $\mu\searrow 0$. Additionally,

$$I_t p\rightharpoonup I_t p^{\mu=0}\text{weaklyin}L^2(0,T;H^1(\Omega))\text{as}\mu\searrow 0.$$

Furthermore, from (58) we have the uniform bound

$$\sqrt{\mu}\|\nabla(\nabla\cdot\boldsymbol{u})\|_{L^2(L^2(\Omega))}\leq C$$

and thus know that

$$\int_0^T\int_\Omega\mu\nabla(\nabla\cdot\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}x\mathrm{d}t\to 0\quad\text{as }\mu\searrow 0.$$

These convergence results allow us to pass to the limit in (52) and prove existence of solutions for the problem without viscosity. To give the statement, we introduce the space

$$X_{\boldsymbol{u}}=\{\boldsymbol{u}\in L^\infty(0,T;H(\mathrm{div};\Omega))\ :\ \boldsymbol{u}\cdot\nu=0\text{ on }\partial\Omega\}.$$

Recall that the spaces X_σ and $X_{\mathrm{I}_t p}$ are defined in (9) and (10), respectively. We can thus state the second main result of this work.

Theorem 2 (Existence of solutions when $\mu=0$). *Under the assumptions of Proposition 5, there exists* $(\boldsymbol{u}^{\mu=0},\sigma^{\mu=0},p^{\mu=0})\in\mathcal{X}=X_{\boldsymbol{u}}\times X_\sigma\times X_{\mathrm{I}_t p}$ *that satisfies*

$$\int_0^T\int_\Omega\Big\{\left(\rho_0(\boldsymbol{u}^{\mu=0}-\boldsymbol{u}_0)+\nabla\mathrm{I}_t p^{\mu=0}-\mathrm{I}_t\boldsymbol{f}\right)\cdot\boldsymbol{v}+\left(\sigma_t^{\mu=0}+a(\sigma^{\mu=0})\nabla\cdot\boldsymbol{u}^{\mu=0}\right)w$$
$$+\left(\mathrm{I}_t p^{\mu=0}-c_0^2\rho_0\mathrm{I}_t(b(\sigma^{\mu=0})\sigma^{\mu=0})\right)\Delta_{1/\rho_0}\phi+2\alpha_0\left(\tau(-\Delta)^{\frac{y}{4}}(\sigma^{\mu=0}-\sigma_0)(-\Delta)^{\frac{y}{4}}\phi+\eta(-\Delta)^{\frac{y+1}{4}}\mathrm{I}_t\sigma^{\mu=0}(-\Delta)^{\frac{y+1}{4}}\phi\right)\Big\}\,\mathrm{d}x\mathrm{d}t$$
$$=0\tag{57}$$

for all $\boldsymbol{v} \in L^2(0,T;L^2(\Omega)^d)$, $w \in L^2(0,T;L^2(\Omega))$, *and* $\phi \in L^1(0,T;H^{\frac{\gamma+1}{2}}(\Omega))$, *such that* $\nabla\phi \cdot \nu = 0$ *with* $\sigma^{\mu=0}|_{t=0} = \sigma_0$. *Furthermore, the following bound holds:*

$$\|\boldsymbol{u}^{\mu=0}\|^2_{L^\infty(H(div;\Omega))} + \|\nabla\sigma^{\mu=0}\|^2_{L^\infty(L^2(\Omega))} + \|\sigma^{\mu=0}\|^2_{L^\infty(H^{\frac{\gamma+1}{2}}(\Omega))} + \int_0^T \left(\|\sigma_t^{\mu=0}(t)\|^2_{H^{\frac{\gamma}{2}}(\Omega)} + \|\nabla \mathrm{I}_t p^{\mu=0}\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$
$$\leq C_1\exp(C_2T)\Big(\|\nabla\cdot(\rho_0^{-1}\boldsymbol{f})\|^2_{L^2(L^2(\Omega))} + \|\mathrm{I}_t\boldsymbol{f}\|^2_{L^2(L^2(\Omega))} + \|\boldsymbol{u}_0\|^2_{H(div;\Omega)} + \|c_0\sqrt{\sigma_0}\,\nabla\sigma_0\|^2_{L^2(\Omega)} + \|\sigma_0\|^2_{H^{\frac{\gamma+1}{2}}(\Omega)}\Big), \tag{58}$$

where the constants C_1 *and* C_2 *do not depend on* μ.

With this result, we have established sufficient conditions for the existence of solutions to (1) with the modified absorption operator (6), where the problem is understood in the sense of (57). As previously mentioned, the theory can also be adapted to allow for having the original absorption operator (2), however at the cost of higher smoothness of the coefficients ρ_0 and c_0.

ACKNOWLEDGMENTS

The contribution of B.K. to this work was supported by the Austrian Science Fund (FWF) (10.55776/P36318). B.C. acknowledges the support of the Engineering and Physical Sciences Research Council, UK (EP/W029324/1, EP/T014369/1). The authors thank the reviewer for the helpful comments that have led to an improved version of the manuscript.

CONFLICT OF INTEREST STATEMENT

The authors have no conflict of interest to declare.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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How to cite this article: Cox B, Kaltenbacher B, Nikolić V, Lucka F. Existence of solutions to k-Wave models of nonlinear ultrasound propagation in biological tissue. *Stud Appl Math*. 2024;e12771. https://doi.org/10.1111/sapm.12771

APPENDIX

We here provide the proof of Lemma 2, which is partly based on the following consequence of the Courant–Fischer max – min formula of eigenvalues of compact self-adjoint operators, adapted from Ref. [20].

Lemma A1. *Let V and H be Hilbert spaces with $C : V \to V$ self-adjoint and compact and $\mathcal{M} : V \to H$ boundedly invertible with $\mathcal{M} \in L(V, H)$ and $\mathcal{M}^{-1} \in L(H, V)$. Then the operator $\tilde{C} := (\mathcal{M}^{-1})^* C \mathcal{M}^{-1} : H \to H$ is self-adjoint and compact and the eigenvalues λ_k of C and μ_k of $\tilde{C}$ decay at the same rate; more precisely, it holds*

$$\frac{1}{\|\mathcal{M}^{-1}\|^2}\mu_k \le \lambda_k \le \|\mathcal{M}\|^2 \mu_k,$$

with $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$ and $\mu_1 \ge \mu_2 \ge \cdots \ge 0$.

Proof. Recall that by the Courant–Fischer Theorem, the eigenvalues in decreasing order obey the following variational characterization:

$$\lambda_k = \max\{\min\{(Cx, x)_V : x \in S_k, \|x\| = 1\} : \dim(S_k) = k, S_k \text{ subspace of V}\}.$$

From this characterization, we obtain

$$\begin{aligned}
\lambda_k &= \max_{\dim(S_k)=k}\ \min_{x\in S_k,\|x\|=1} (Cx,x)_V \\
&= \max_{\dim(S_k)=k}\ \min_{x\in S_k,\|x\|=1} ((\mathcal{M}^{-1})^* C\mathcal{M}^{-1}\,\mathcal{M}x,\mathcal{M}x)_V \\
&= \max_{\dim(S_k)=k}\ \min_{x\in S_k,\hat{x}=\mathcal{M}x/\|\mathcal{M}x\|,\|x\|=1} (\tilde{C}\hat{x},\hat{x})_H \|\mathcal{M}x\|^2 \\
&\geq \frac{1}{\|\mathcal{M}^{-1}\|} \max_{\dim(S_k)=k}\ \min_{x\in S_k,\hat{x}=\mathcal{M}x/\|\mathcal{M}x\|,\|x\|=1} (\tilde{C}\hat{x},\hat{x})_H = (\star),
\end{aligned}$$

using $\|\mathcal{M}x\| \geq \frac{1}{\|\mathcal{M}^{-1}\|}\|x\| = \frac{1}{\|\mathcal{M}^{-1}\|}$. Due to the fact that

$$\hat{x} = \frac{\mathcal{M}x}{\|\mathcal{M}x\|} \in \hat{S}_k = \mathcal{M}S_k,$$

and the dimension of S_k being k, due to regularity of $\mathcal{M}$, $\hat{S}_k$ is of dimension k as well. Therefore, taking the minimum over a superset by dropping the constraint $\|x\| = 1$ results in

$$(\star) \geq \frac{1}{\|\mathcal{M}^{-1}\|^2} \max_{\dim(\hat{S}_k)=k}\ \min_{\hat{x}\in\hat{S}_k,\|\hat{x}\|=1} (\tilde{C}\hat{x},\hat{x})_H = \frac{1}{\|\mathcal{M}^{-1}\|^2}\mu_k.$$

Analogously, it holds

$$\mu_k \geq \frac{1}{\|(\mathcal{M}^{-1})^{-1}\|^2}\lambda_k = \frac{1}{\|\mathcal{M}\|^2}\lambda_k,$$

which concludes the proof. □

Proof of Lemma 2.

Proof. The first estimate in (25) follows by testing (24) with $v^n = \mathrm{P}^{\rho_0}_{W^n}\mathfrak{g}$ and using the Cauchy–Schwarz inequality as well as the estimate

$$\|\nabla \mathrm{g}(\boldsymbol{u}^n)\|_{L^2(\Omega)} \leq \|\nabla\nabla \ln \rho_0\|_{L^3(\Omega)}\|\boldsymbol{u}^n\|_{L^6(\Omega)} + \|\nabla \ln \rho_0\|_{L^\infty(\Omega)}\|\nabla \boldsymbol{u}^n\|_{L^2(\Omega)},$$

where $\nabla\nabla$ denotes the Hessian. For the second and third bounds in (25), we recall the definition of the fractional power of a symmetric nonnegative operator A with eigensystem $\{(\lambda_i, w_i)\}_{i\geq 1}$ as

$$A^\gamma v = \sum_{i\in\mathbb{N}} \lambda_i^\gamma (v, w_i). \tag{A1}$$

We also note that

$$\|(-\Delta_N)^\gamma \mathrm{P}^{\rho_0}_{W^n}\mathfrak{g}\|_{L^2(\Omega)} \leq \|\rho_0\|^\gamma_{L^\infty(\Omega)}\|(-\Delta_{1/\rho_0})^\gamma \mathrm{P}^{\rho_0}_{W^n}\mathfrak{g}\|_{L^2(\Omega)}, \quad \gamma \in \left\{\frac{y}{4}, \frac{y+1}{4}\right\}.$$

We test (24) with

$$v^n = (-\Delta_{1/\rho_0})^{\gamma-\frac{1}{2}} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g} \in W^n \quad \text{for } \gamma \in \left\{ \frac{y}{4}, \frac{y+1}{4} \right\}$$

to obtain

$$\|(-\Delta_{1/\rho_0})^{\gamma} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \le \|(-\Delta_{1/\rho_0})^{\gamma} \mathfrak{g}\|_{L^2(\Omega)}.$$

Then we make use of Lemma A1 with $V = H = L^2(\Omega)$, $C = (-\Delta_{1/\rho_0})^{-1}$, $\tilde{C} = (-\Delta_N)^{-1}$, $\mathcal{M} = (-\Delta_N)^{1/2}(-\Delta_{1/\rho_0})^{-1/2}$, and

$$\begin{aligned}
\|\mathcal{M}\| &= \sup_{v \in L^2(\Omega)\setminus\{0\}} \frac{\|(-\Delta_N)^{1/2}(-\Delta_{1/\rho_0})^{-1/2} v\|_{L^2(\Omega)}}{\|v\|_{L^2(\Omega)}} \\
&= \sup_{w \in H^1_{\diamond}(\Omega)\setminus\{0\}} \frac{\|(-\Delta_N)^{1/2} w\|_{L^2(\Omega)}}{\|(-\Delta_{1/\rho_0})^{1/2} w\|_{L^2(\Omega)}} \\
&= \sup_{w \in H^1_{\diamond}(\Omega)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega)}}{\|\sqrt{\frac{1}{\rho_0}} \nabla w\|_{L^2(\Omega)}} \le \|\rho_0\|_{L^\infty(\Omega)},
\end{aligned}$$

where $H^1_{\diamond}(\Omega)$ denotes the space of zero mean functions in $H^1(\Omega)$. Likewise $\|\mathcal{M}^{-1}\| \le \|\frac{1}{\rho_0}\|_{L^\infty(\Omega)}$. Using (A1), we thus obtain

$$\|(-\Delta_{1/\rho_0})^{\gamma} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \le \|(-\Delta_{1/\rho_0})^{\gamma} \mathfrak{g}\|_{L^2(\Omega)}.$$

Combining the bounds leads to

$$\begin{aligned}
\|(-\Delta_N)^{\gamma} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} &\le \|\rho_0\|^{\gamma}_{L^\infty(\Omega)} \|(-\Delta_{1/\rho_0})^{\gamma} \mathrm{P}^{\rho_0}_{W^n} \mathfrak{g}\|_{L^2(\Omega)} \\
&\le \|\rho_0\|^{\gamma}_{L^\infty(\Omega)} \|(-\Delta_{1/\rho_0})^{\gamma} \mathfrak{g}\|_{L^2(\Omega)} \le \|\rho_0\|^{\gamma}_{L^\infty(\Omega)} \left\| \frac{1}{\rho_0} \right\|^{\gamma}_{L^\infty(\Omega)} \|(-\Delta_N)^{\gamma} \mathfrak{g}\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, we apply the Kato–Ponce–type estimate to further infer

$$\|(-\Delta_N)^{\gamma} \mathfrak{g}\|_{L^2(\Omega)} = \|(-\Delta_N)^{\gamma}[\nabla \ln \rho_0 \cdot \boldsymbol{u}^n]\|_{L^2(\Omega)} \lesssim \|\nabla \ln \rho_0\|_{L^\infty(\Omega)} \|\boldsymbol{u}^n\|_{H^{2\gamma}(\Omega)} + \|\nabla \ln \rho_0\|_{H^{2\gamma}(\Omega)} \|\boldsymbol{u}^n\|_{L^\infty(\Omega)},$$

from which then the second and third estimates in (25) follow. □